GROTHENDIECK
THE MATHEMATICIAN

A TIME FOR REFLECTIONS 1973–1991

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A TIME FOR REFLECTIONS
1973–1991



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Image sources

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- Jean Malgoire private collection

Invitation to travel

Heureux celui qui peut d'une aile vigoureuse,
S'élancer vers les champs lumineux et sereins;
Charles Baudelaire,
Les Fleurs du mal, "Élévation"

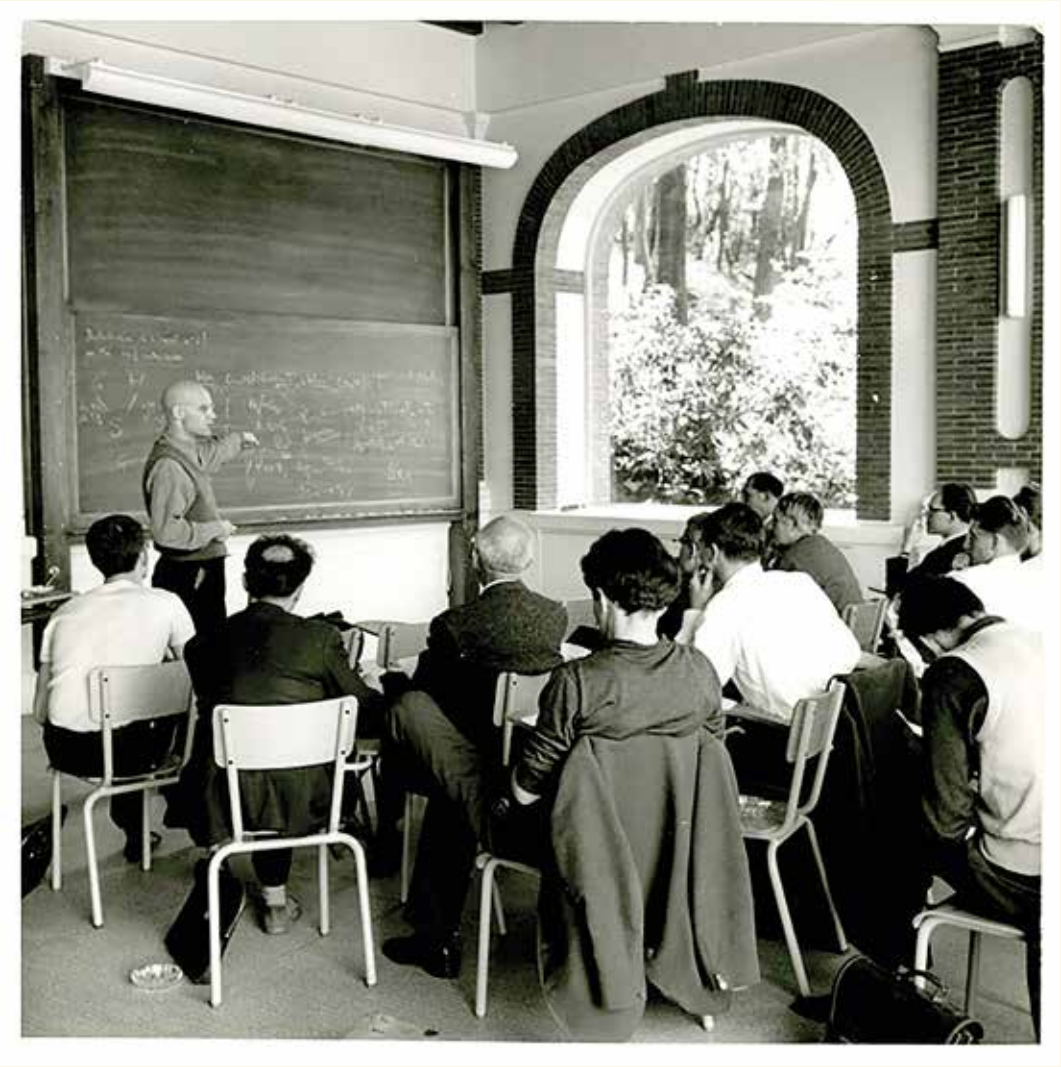
There was a mathematician who, at the height of his fame, abandoned scientific research to advocate radical ecology, became a professor at the University of Montpellier, devoted himself to meditation, was regarded by some as a prophet and by others as hotheaded, and then disappeared. One could use these episodes to construct the biography of a "frightening genius" and write stories embellished with anecdotes and colorful portraits likely to entertain *amateurs* of famous scientists.

Our approach is different. In this exhibition, we wanted to compose a portrait of Alexandre Grothendieck, the mathematician, placing ourselves as far as possible from the storms of his biography.

*

However, traces of his unceasing mathematical research between 1973 and 1991 are both visible and numerous, as evidenced by the thousands of pages written by Grothendieck between his arrival in Montpellier and his departure for the village of Lasserre in Ariège. These pages, entrusted to Jean Malgoire and now available on the website of the Grothendieck Archives at the University of Montpellier, provided most of the documentation and images for this exhibition.

These texts provide the constantly renewed material for "mathematical reflections," of which *La Longue Marche à travers la théorie de Galois* (1981), *Pursuing Stacks* (1983) and *Esquisse d'un Programme* (1984) are examples that we have examined. As for *Récoltes et Semailles* (1984-1986), in which Grothendieck once again offers "reflections" but also a "testimony about a mathematician's past," it came after these samples and ended up replacing them and those announced but never published.



Bouillot, “Alexandre Grothendieck au séminaire de géométrie algébrique,”
© Archives de l’IHES

It has been argued that the baroque and provincial mathematics written by Grothendieck in a "renewed style" between 1973 and 1991 formed the tail of a comet observed in all its glory between 1955 and 1970: the comet of algebraic geometry of schemes and topos, classical and sophisticated, written in the formal style of a member of Bourbaki.

This judgement is contradicted by the mathematical texts of this period and their outcome, but also by Grothendieck himself in his retrospective reflection on "arithmetic geometry" and "topological algebra." Indeed, these new sciences are developed, according to a necessary balance, over the two periods. As for his style of mathematical expression, while he concedes that it changed and conformed in *Pursuing Stacks* to that of a "logbook," Grothendieck did not reject "formal writing" subject to the venerable canon of the axiomatic-deductive method.

*

We are aware of the difficulties of reading an exhibition that focuses on Grothendieck as a mathematician. It is true that it is aimed more at people with a university education in mathematics. We do not claim to solve this problem inherent in the dissemination of mathematics. Despite the defects of this first attempt, we still believe that a complex analysis of facts worthy of interest is more respectful to the reader than simplified and therefore misleading formulas.

We hope that, despite these difficulties, readers will enjoy following with us some fragments of the mathematical exploration formerly undertaken by Grothendieck. The territories of this new continent are the mathematical theories he discovered, and the stages of this journey are the texts he wrote.

Sébastien Maronne, Bertrand Toën
Toulouse, November 2025 (English version)

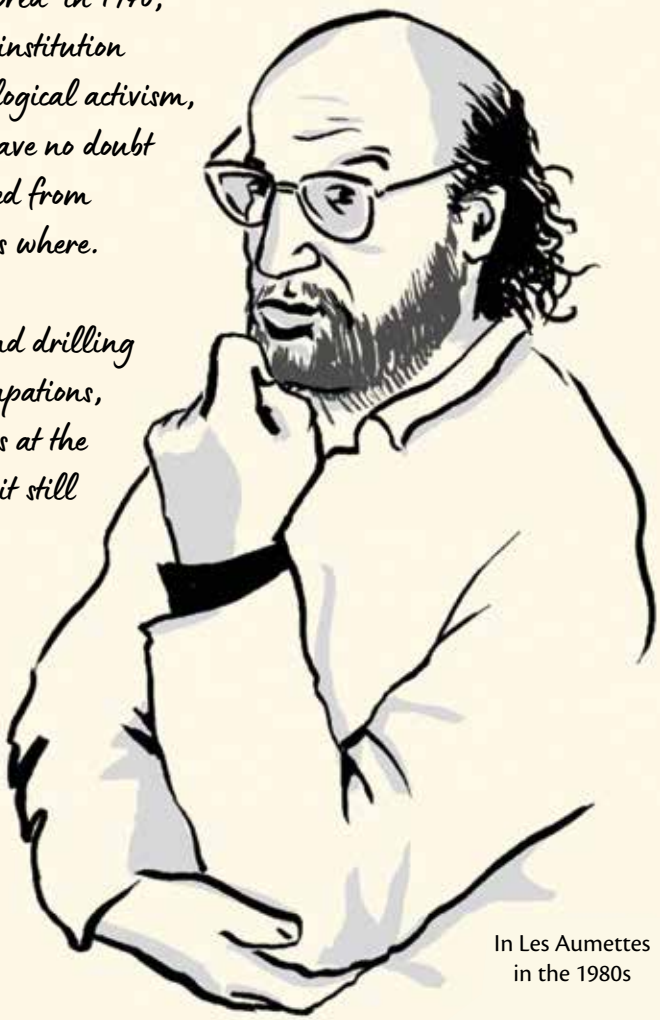
Alexandre GROTHENDIECK

Elements of a biography

“As you know, I left the mathematical ‘big world’ in 1970, following a story about military funds at my home institution (IHES). After a few years of anti-militarist and ecological activism, in the style of the ‘cultural revolution’, which you have no doubt heard about here and there, I practically disappeared from circulation, lost in a provincial university God knows where.

Rumor has it that I spend my time herding sheep and drilling wells. The truth is that, apart from many other occupations, I bravely went, like everyone else, to teach my classes at the university (that was my unoriginal livelihood, and it still is today).

Every now and then, for a few days, even a few weeks or months, I would even go back to doing maths “according to my whims [à brin de zinc]” – I have boxes full of my scribbles, which I must be the only one who can decipher. But these were very different things, at least at first glance, from what I had done in the past.



In Les Aumettes in the 1980s

Between 1955 and 1970, my favorite subject had been cohomology, and more specifically, the cohomology of varieties of all kinds (algebraic, in particular). I felt I had done enough in that direction for others to manage without me, and if I was going to do maths, it was time to change the tune...

Récoltes et Semailles, “Une lettre,” May 1985, p. L3

1928: Birth in Berlin

1930

A. Grothendieck at Pont-à-Mousson in 1951



1945 - 1948: Bachelor's degree in mathematics at the University of Montpellier

1940

1948 - 1949: Auditor at the École Normale Supérieure in Paris

1949 - 1953: Research in Nancy under the supervision of J. Dieudonné and L. Schwartz

1950

1953 - 1955: University of São Paulo (Brazil) then University of Kansas (USA)

1953: Doctoral thesis *Produits tensoriels topologiques et espaces nucléaires*

1957: *Sur quelques points d'algèbre homologique*

1958: *The cohomology theory of abstract algebraic varieties*

1956 - 1958: Researcher at the CNRS

1960 - 1967: *Éléments de Géométrie Algébrique (EGA)*

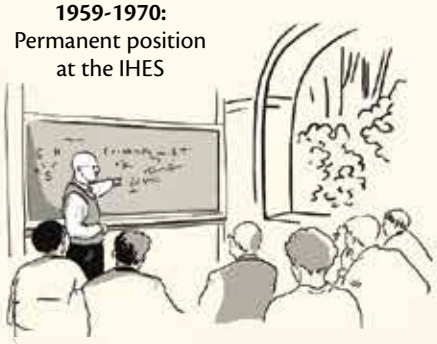
1960

1968: *Standard conjectures on algebraic cycles*
Dix exposés sur la cohomologie des schémas

1970

1966: Fields Medal

1959-1970: Permanent position at the IHES



1981: *La Longue Marche à travers la théorie de Galois*

1980

1960-1977: *Séminaire de Géométrie Algébrique (SGA)*

1st October 1970: Resignation from IHES

1983: *Pursuing Stacks*

1973 - 1984: Professor at the University of Montpellier

1970 - 1973: Visiting Professor at the Collège de France, then at Orsay University

1984: *Esquisse d'un Programme*

1984 - 1986: *Récoltes et Semailles*

1984 - 1988: Researcher at the CNRS

1988: Retirement / Refusal of the Crafoord Prize

1986: *Vers une Géométrie des formes*

1990

1990 - 1991: *Les Dérivateurs*

2000

1991: Moving to Lasserre

2014: Death at Saint-Girons Hospital

2010

Alexandre Grothendieck's places

LES AUMETTES 1980-1991

LA GARDETTE 1979-1980

Université de Montpellier 1973-1984

VILLECUN 1973-1979

LASSERRE 1991-2014

Mediterranean Sea

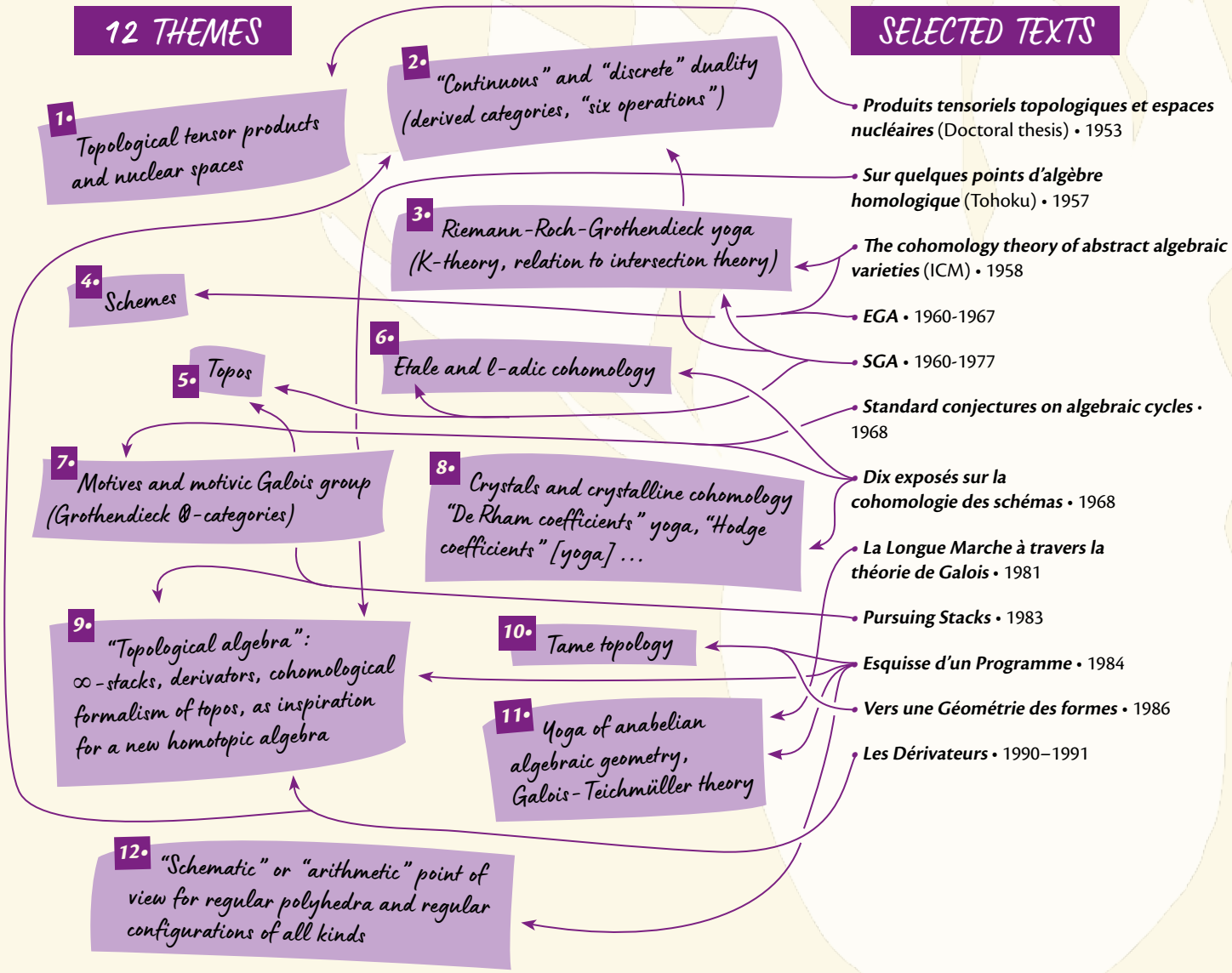
The Pyrenees

THEMES

Among the many new perspectives I have developed in mathematics, there are twelve that, in retrospect, I would call 'great ideas'.

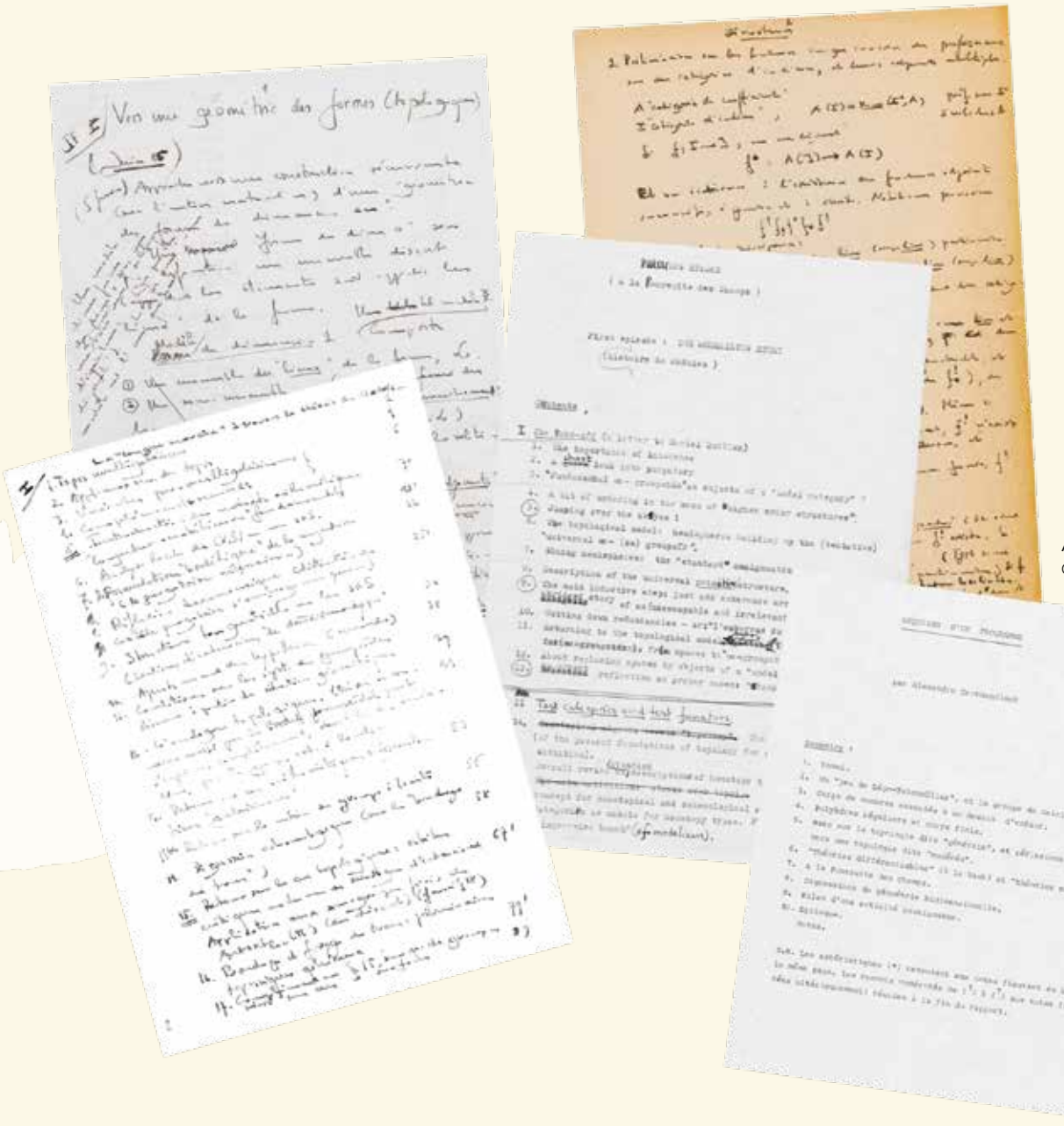
Récoltes et Semailles, "Promenade à travers une œuvre," January 1986, p. P21

In *Récoltes et Semailles*, Grothendieck took stock of his mathematical work and listed more particularly **twelve themes** or **"great ideas"**:



"There are three [themes 10-11-12] (and not the least important in my view) which, since they have appeared only after my departure from the mathematical scene, remain in an embryonic state, 'officially' they do not even exist, since there is no formal publication to serve as their birth certificate.

Récoltes et Semailles, "Promenade à travers une œuvre," January 1986, pp. P21-22



Archives Grothendieck
© Université de Montpellier

THE "MONTPELLIER" PERIOD

The period of Grothendieck's mathematical work, between **1955** and **1970**, is well known. It was a golden age for algebraic geometry, with the discovery of schemes, topos, etc., and the publication of thousands of pages of *EGA* and *SGA*.

The "Montpellier" period studied here is less well known. However, the mathematical work accomplished between **1973** and **1991** occupies a significant portion of the 18,000 pages of archives digitized by the University of Montpellier. These have been available online since 2017. Furthermore, Grothendieck often emphasized the profound continuity between these two periods of mathematical activity.

The last three **themes, 10 to 12**, were only developed after 1973, while **theme 9** bridged the two periods. These themes, presented in *Esquisse d'un Programme* in 1984, were then the subject of informal communications between mathematicians. Some of Grothendieck's **visions** were thus realized by the mathematical community. Others continue to be explored today.

ARITHMETIC GEOMETRY

Let V be a variety without singular points, of dimension n, defined over a finite field k with q elements. Let N_r be the number of rational points on V over the extension k_r of k of degree r. Then we have

$$\sum_1^\infty N_r U^{r-1} = \frac{d}{dU} \log Z(U),$$

where $Z(U)$ is a rational function in U, satisfying a functional equation

$$Z\left(\frac{1}{q^n U}\right) = \pm q^{n\chi/2} U^\chi Z(U),$$

with χ equal to the Euler-Poincaré characteristic of V (intersection-number of the diagonal with itself on the product $V \times V$).

Furthermore, we have:

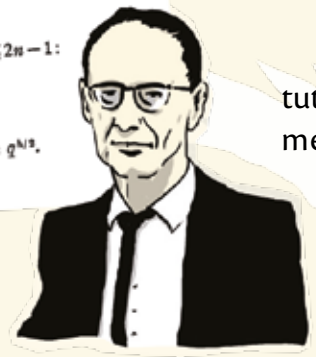
$$Z(U) = \frac{P_1(U) P_3(U) \cdots P_{2n-1}(U)}{P_0(U) P_2(U) \cdots P_{2n}(U)},$$

with $P_0(U) = 1 - U$, $P_{2n}(U) = 1 - q^n U$, and, for $1 \leqq h \leqq 2n-1$:

$$P_h(U) = \prod_{i=1}^{B_h} (1 - \alpha_{hi} U)$$

where the α_{hi} are algebraic integers of absolute value $q^{h/2}$.

The statement of Weil's conjectures in the article "Number of solutions of equations in finite fields" (1949)



André Weil (1906-1998)

WEIL'S CONJECTURES

André Weil published his famous conjectures on algebraic varieties with coefficients in a finite field in 1949. By progressing in the algebraic geometry of finite fields, the true "hub" of a "**trilingual text**" that also contains complex analytical geometry and number theory, the aim is to complete the middle column of this still incomplete text, with the **Riemann hypothesis** as a backdrop.

> *Since Artin's thesis, German school algebraists had been exploring the analogies between number fields and function fields over finite fields, to this diptych I had added a third section for my own use, classical Riemannian theory [...] I liked to see this as a 'trilingual text' (the 'Rosetta Stone' [...]) whose three columns were meant to illuminate each other, my main purpose being to fill in the gaps.*
>
> André Weil, *Œuvres Scientifiques* I, p. 542

A NEW GEOMETRY

For Grothendieck, Weil's conjectures constituted "the first embryo of this vision of 'arithmetic geometry'" because Weil emphasized **topological invariants** in his statements. The fundamental notions of this new geometry were those of **schemes** and **topos**, which generalized the classical notions of algebraic variety and space.

These new concepts enabled Grothendieck to develop cohomological theories capable of proving Weil's conjectures. He only succeeded in part. **Pierre Deligne** proved the analogue of Riemann's hypothesis, considered the most difficult in 1974, by incorporating new ideas.

> *[These] absolutely breathtaking conjectures, to tell the truth, [...] suggested, for these new 'varieties' (or 'spaces') of a discrete nature, the possibility of certain types of constructions and arguments that until now seemed conceivable only in the context of the 'spaces' considered worthy of the name by analysts – namely, the so-called 'topological' spaces [...]*
>
> *Récoltes et Semailles*, "Promenade à travers une œuvre," January 1986, p. P30

The vision was that of an 'arithmetic geometry', a synthesis of topology, geometry (algebraic and analytical), and arithmetic, the first seeds of which I found in Weil's conjectures.

Récoltes et Semailles, "Une lettre," May 1985, p. L6

STANDARD CONJECTURES ON ALGEBRAIC CYCLES

By A. GROTHENDIECK

1. **Introduction.** We state two conjectures on algebraic cycles, which arose from an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties. These are not really new, and they were worked out about three years ago independently by Bombieri and myself.

The statement of the standard conjectures in the article "Standard conjectures on algebraic cycles" (1968)

More ambitiously, Grothendieck devoted himself to the dream of **motives** that were supposed to constitute the "common denominator" of such cohomologies. In 1968, he formulated the "**standard conjectures,**" which were more general but, above all, according to him, more geometric and therefore in line with his vision.

GEOMETRY OR NUMBER THEORY?

According to Grothendieck, arithmetic geometry had to bring together the hitherto separate domains of (integer) numbers and (continuous) magnitudes on the basis of a new theory of shape. Hence the need for a **topological algebra**, "logistical support" in the development of this new geometry.

This vision was quite different from that of Weil, who emphasized the fruitfulness of mysterious analogies between separate and hierarchized domains and did not intend to subject number theory to geometry.

In 1978, Deligne renamed his seminar at IHES "arithmetic geometry seminar." In doing so, he wanted to emphasize arithmetic and make room for tools other than those derived from Grothendieck's "machine," such as those stemming from **Langlands' program**.

"A 'SECOND NATURE' THAT IS STILL ALIVE IN ME!"

The vision of arithmetic geometry pervaded the mathematics of the Montpellier period. It reappeared with the themes of **anabelian geometry** and **Galois-Teichmüller theory**, which Grothendieck developed from 1977 onwards. These themes constituted a key part of arithmetic geometry alongside the theory of motives.

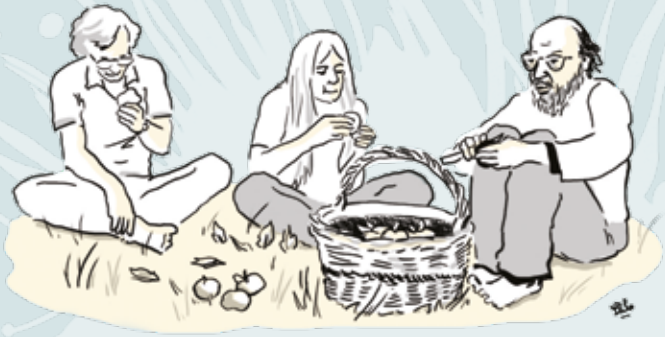
Research report (1st October-15 December 1984),
Archives Grothendieck, © Université de Montpellier

depuis 1978. En termes imagés, on peut dire que ces deux courants d'idées, le courant "abélien" incarné par la notion de motif, et le courant "anabélien" exemplifié par la structure géométrico-arithmétique de la "tour de Teichmüller", sont à la "géométrie arithmétique" dans son enfance, ce que les courants "complexes de cochaines - catégories dérivées commutatives" et "champs en tous genres - catégories homotopiques non commutatives" sont à l'algèbre topologique (encore in utero).
La différence essentielle entre ces deux disciplines ne se situe cependant nullement dans leur état d'avancement relatif, circonstance des plus contingentes, apte à changer radicalement en l'espace de quelques années, ne serait-ce que par la vertu de l'écriture de "Pursuing Stacks". Elle se situe plutôt dans une différence de profondeur. Comme c'était

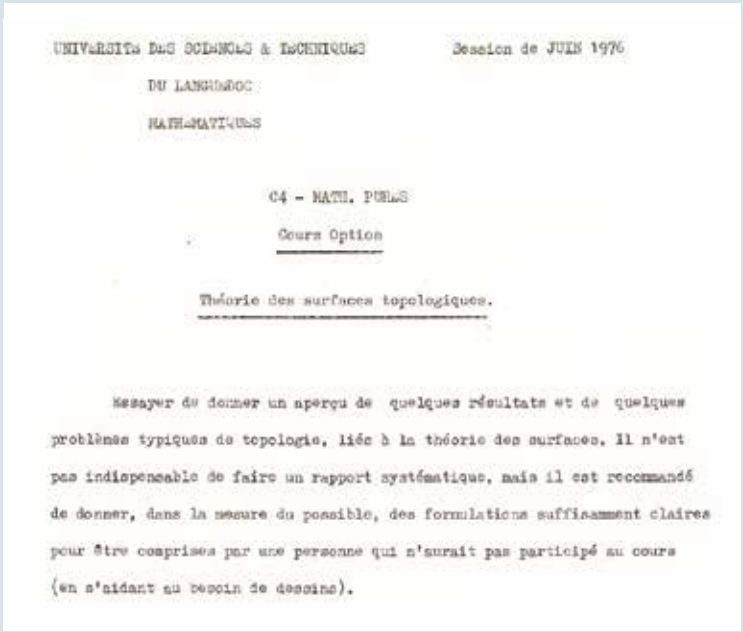
A "RESEARCH-BASED TEACHING"

PROFESSOR IN MONTPELLIER

Grothendieck was appointed professor at the University of Montpellier at the start of the 1973 academic year. He proposed there an original teaching method grounded in the basic **intuition** of surfaces with deliberately modest prerequisites. This led him to renew his own research themes by focusing on elementary objects rather than on objects of **maximum generality**, as he had done previously.



The mathematical déjeuner sur l'herbe (1979), illustration after a photograph by Jean Malgoire



UNIVERSITE DES SCIENCES & TECHNIQUES Session de JUIN 1976
DU LANGUEDOC
MATHEMATIQUES

C4 - MATH. PURES
Cours Option

Théorie des surfaces topologiques.

Essayer de donner un aperçu de quelques résultats et de quelques problèmes typiques de topologie, liés à la théorie des surfaces. Il n'est pas indispensable de faire un rapport systématique, mais il est recommandé de donner, dans la mesure du possible, des formulations suffisamment claires pour être comprises par une personne qui n'aurait pas participé au cours (en s'aidant au besoin de dessins).

The exam subject for the master's course "*Théorie des surfaces topologiques*" (1976), © Université de Montpellier

STUDENTS

Students accompanied Grothendieck each time he returned to mathematical research. In *Esquisse d'un Programme*, he counted about twenty students to whom he strove to impart the "rudiments of the mathematical profession" with great rigor.

A small group thus devoted itself to the mathematical topic of the moment in his office before having lunch together. Grothendieck was usually the speaker and found it difficult to put himself in the position of the listener.

Grothendieck also saw his students as collaborators. While supervising **Yves Ladegaillerie**'s doctoral thesis from 1974 to 1976, he learned key results of surface topology from him, while encouraging him to develop some "typical *Grothen-*

"Rather than courses in the usual sense of the term, these are reflection sessions, conducted jointly with students on topics of their choice, which each student explores in depth through personal work throughout the year. There are no prior restrictions on the topics that can be explored [...], to the point where we are often completely taken aback by situations in which students often have from the outset or develop along the way a very sure, even profound, intuition. Such situations, clearly perceived by all, are a welcome shake-up of the strict "active-passive" roles that are the norm in teaching institutions.

This pedagogical vision has not always been realized, as admitted by the protagonists themselves...

"Research report" Archives Grothendieck, © Université de Montpellier

It seemed important to me to start from a common intuitive background...

Esquisse d'un Programme, January 1984, p. 2

dieckeries." He also gave foundation works related to *topological maps* to **Jean Malgoire** and **Christine Voisin** in a *DEA* (Master's degree) in 1976 and then a PhD thesis defended in 1979.

REGULAR POLYHEDRA

One of the privileged themes taught by Grothendieck as an introduction to research concerned **regular polyhedra**. He was particularly interested in the icosahedron.

In parallel, he devoted some of his research works to a general theory of regular polyhedra as geometric realizations of combinatorial maps over a base ring.



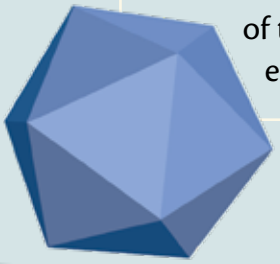
Cours DEA 1977/78 de A. Grothendieck

Géométrie arithmétique des polygônes réguliers

Ce "cours" se fera dans l'esprit d'une investigation en commun de multiples facettes de la notion de polygône régulier au-dessus d'un corps de base, puis d'un anneau de base quelconque. Ce sera une occasion stimulante pour nous familiariser avec les notions de base de géométrie vectorielle, affine, projective - avec ou sans métrique - en dimensions un, deux et trois, et avec les phénomènes particuliers à ces dimensions, ainsi qu'avec les aspects parfois imprévus que prennent des notions familières (comme celle de réflexion, ou de centre d'un polygône régulier) sur les corps de caractéristique positive, et notamment sur les corps finis. Le besoin d'une géométrie sur des anneaux de base généraux (tel l'anneau "sur lequel s'écrit" le n-polygône régulier "le plus général") se fera sentir en cours de route, et nous amènera sans doute à affleurer quelques points de fondements de géométrie algébrique. Si le temps le permet et si d'autres investigations ne nous retiennent , nous tenterons, dans la volée, de construire l'icosaèdre régulier "général" sur un anneau de base convenable, fini sur l'anneau $\mathbb{Z}$ des entiers, ayant des corps résiduels de toutes caractéristiques. (La caractéristique deux donnera sans doute du fil à retordre !).

Master's course "*Géométrie arithmétique des polygones réguliers*" (1977-78),
Archives Grothendieck © Université de Montpellier

The regular icosahedron is one of the five Platonic solids. The twenty faces of this regular polyhedron are equilateral triangles.



RECOGNITION OF FAILURE

When writing *Esquisse d'un Programme* in January 1984, Grothendieck "clear[ly] and unequivocal[ly]" recognized that his research teaching at Montpellier had failed. He attributed this failure to his own inability to listen and communicate, and to the mathematical training of his students, which he considered too modest.

This latent **isolation** was accentuated by his secondment to the CNRS in 1984 and his departure from the University of Montpellier. This marked the beginning of a period of solitary work, punctuated by correspondence resulting from the publication of *Récoltes et Semailles*, during which Grothendieck gradually distanced himself from mathematics.

MATHEMATICAL REFLECTIONS

PRIVATE REFLECTIONS

Despite competing interests, including meditation from October 1976 onwards, Grothendieck never entirely abandoned research and devoted himself sporadically and compulsively to private mathematical reflections. He aimed to prospect the richness of the mathematical world.

> *"The obvious way to 'resolve' such frustration [...] is to do at least some of these things myself, which I despaired of seeing any of my students tackle in the end. This is what I have done here and there, whether through occasional reflection lasting a few hours or even a few days, on the sidelines and during my teaching activities, or during periods of intense mathematical craving (which sometimes occurred like real explosions...), which could last for weeks or months.*
>
> *Récoltes et Semailles*, March 1984, p. 133

CONTINUITY AND RENEWAL

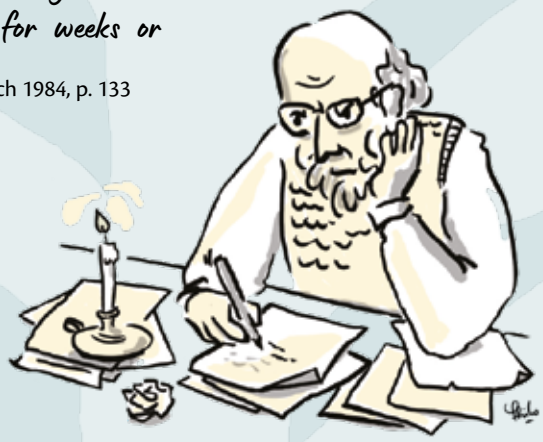
Grothendieck's new **themes of reflection** originated in topology of surfaces, but they remained deeply linked to the themes of his mathematical past, as he noted in hindsight.

Arithmetic geometry was a "new science" which integrated anabelian geometry and Galois-Teichmüller theory, as well as what he called "yoga of motives," while homotopical algebra and tame topology complemented homological algebra and topos theory within a **topological algebra** that served as language and tool for this science.

ESQUISSE D'UN PROGRAMME

The *Esquisse d'un Programme*, written by Grothendieck in January 1984 for a CNRS application, was intended to describe the various mathematical themes he had addressed in previous years.

Grothendieck wanted to develop and **communicate** the visions he had set down in thousands of handwritten pages, believing himself to be the only one capable of deciphering what he described as "**scribbles**" or "***scratchworks.***"



These two volumes are the first in a planned, considerably wider series of "Réflexions Mathématiques" in which the author intends to present, among other things, some "naive ideas" about two "new continents", eager to be discovered and explored. One stems from

Excerpt from a presentation text for *Réflexions mathématiques* copied by Ronnie Brown, December 1983,
Archives Grothendieck © Université de Montpellier

> *… a comprehensive draft, at least for the time being, intended for the mathematical community.*
>
> *Esquisse d'un Programme*, January 1984, p. 1

MATHEMATICAL REFLECTIONS

The aim of *Réflexions mathématiques* was precisely to publish these "**key ideas**" from the present and the past in a series of several volumes. Indeed, Grothendieck lacked a contact and relay, as **Jean-Pierre Serre** once was.

The aim was to pass on these ideas to mathematicians "in the loop" who would be able to carry them out and accomplish the major tasks of **laying foundations** that Grothendieck now only wanted to suggest or sketch.

In the spring of 1983, the first volume of *Pursuing Stacks*, "The Modelizing Story," gave the initial impetus to the project.

> *"Also, the idea has been in the air for the last few months – since I decided to publish these informal notes on stacks or whatever it will turn out to be – that I may well go on in the same way, writing up and publishing informal notes on other topics, including tame topology and anabelian algebraic geometry.*
>
> *Pursuing Stacks*, 27 May 1983, p. 194

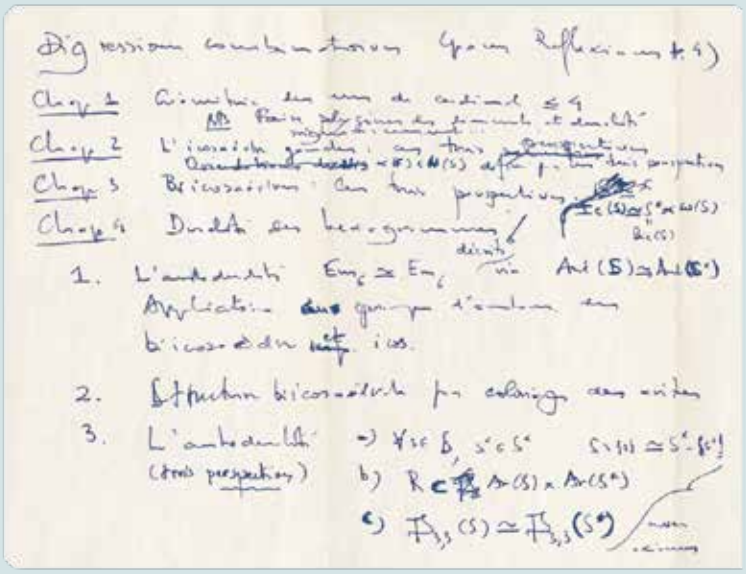
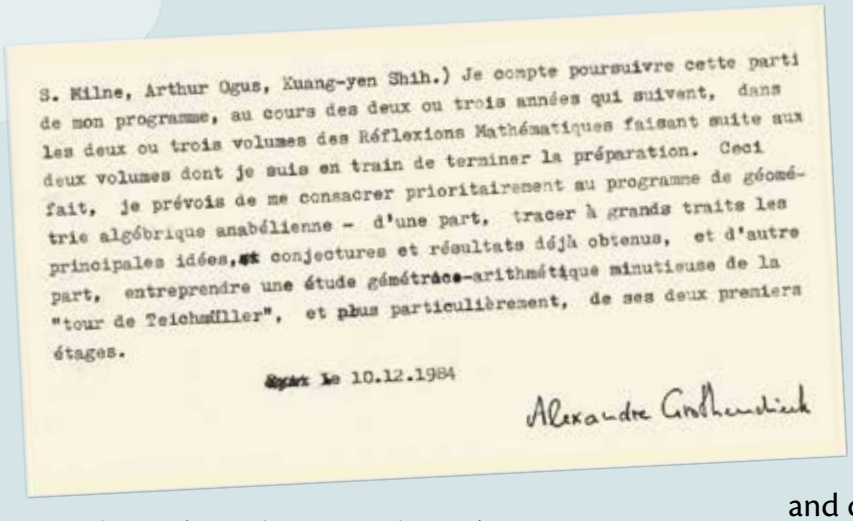


Table of contents of *Digressions combinatoires* announced for volume 4 of *Réflexions mathématiques*, Archives Grothendieck, © Université de Montpellier

AN UNFULFILLED PROJECT

In June 1983, Grothendieck wrote four paragraphs of a more personal introduction to *Pursuing Stacks*. These were the first lines written of ***Récoltes et Semailles***, which would occupy Grothendieck from February 1984 until the spring of 1986.

The outline of *Réflexions mathématiques* continued to evolve as this "testimony to a mathematician's past" grew exponentially, while the initially planned and constantly postponed texts of **mathematical reflection** never saw the light of day.



S. Milne, Arthur Ogus, Kuang-yen Shih.) Je compte poursuivre cette parti de mon programme, au cours des deux ou trois années qui suivent, dans les deux ou trois volumes des Réflexions Mathématiques faisant suite aux deux volumes dont je suis en train de terminer la préparation. Ceci fait, je prévois de me consacrer prioritairement au programme de géométrie algébrique anabélienne – d'une part, tracer à grands traits les principales idées, et conjectures et résultats déjà obtenus, et d'autre part, entreprendre une étude géométrico-arithmétique minutieuse de la "tour de Teichmüller", et plus particulièrement, de ses deux premiers étages.

le 10.12.1984

Alexandre Grothendieck

Research report (1st October–15 December 1984), Archives Grothendieck, © Université de Montpellier

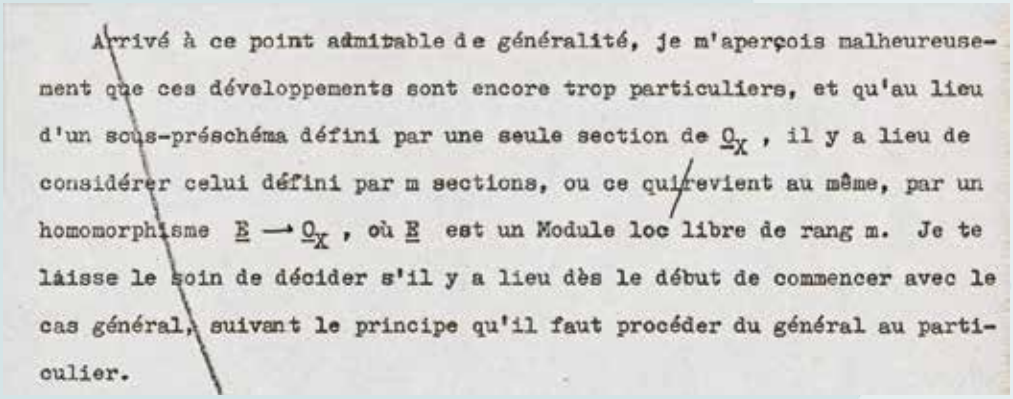
A NEW STYLE OF EXPRESSION?

THE BOURBAKI STYLE



Jean Dieudonné (1906–1992)

Grothendieck completed his mathematical training under the supervision **of Henri Cartan, Jean Dieudonné** and **Laurent Schwartz**, who were members of the **Bourbaki** group. His mathematical texts published between 1955 and 1970, such as *Éléments de Géométrie Algébrique*, written in collaboration with Dieudonné, bore the mark of Bourbaki's writing style.



Arrivé à ce point admirable de généralité, je m'aperçois malheureusement que ces développements sont encore trop particuliers, et qu'au lieu d'un sous-préschéma défini par une seule section de $\underline{O}_X$, il y a lieu de considérer celui défini par m sections, ou ce qui revient au même, par un homomorphisme $\underline{E} \longrightarrow \underline{O}_X$, où $\underline{E}$ est un Module loc libre de rang m. Je te laisse le soin de décider s'il y a lieu dès le début de commencer avec le cas général, suivant le principe qu'il faut procéder du général au particulier.

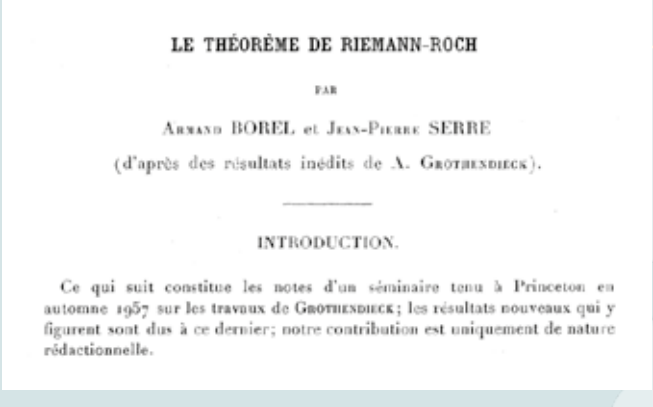
Éléments de Géométrie Algébrique, "New EGA V" (unpublished), typescript annotated by Grothendieck for Dieudonné, Archives Grothendieck
© Université de Montpellier

The mathematicians of the Bourbaki group claimed Hilbert's **axiomatic method** as their model. They adopted a systematic and formal presentation of mathematical concepts, proceeding from the general to the particular, and rarely revealed the motivations behind their choices. The impersonal nature of this style was accentuated by the succession of **drafts** and **collective** critical readings.

Grothendieck left the Bourbaki group in the late 1950s. In *Récoltes et Semailles*, he claimed to have participated in a somewhat distant manner and criticized the "rigid 'canon' of a purely superficial 'rigor'."

The collective nature of mathematics during this period is embodied in the notion of **write-up**. Grothendieck sometimes entrusted his research results to other mathematicians in the form of fairly detailed notes. These provided the material for a seminar, which then led to a formal publication.

Armand Borel and Jean-Pierre Serre, "Le théorème de Riemann-Roch" (1958)



LE THÉORÈME DE RIEMANN-ROCH

PAR

ARMAND BOREL et JEAN-PIERRE SERRE

(d'après des résultats inédits de A. GROTHENDIECK).

INTRODUCTION.

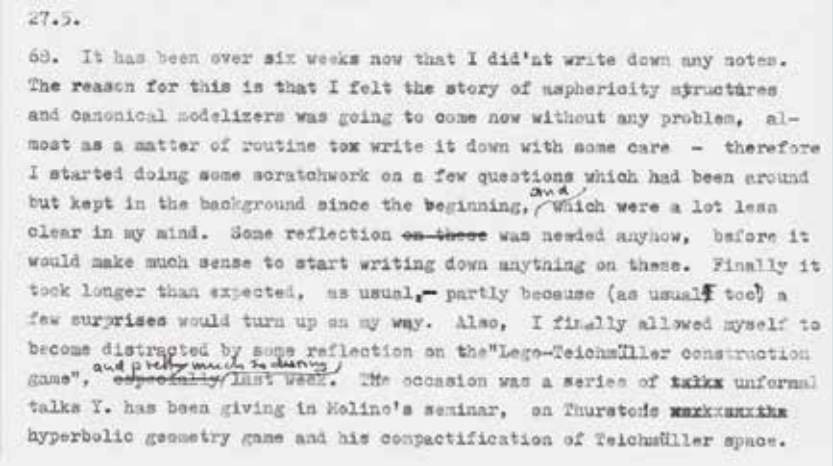
Ce qui suit constitue les notes d'un séminaire tenu à Princeton en automne 1957 sur les travaux de GROTHENDIECK; les résultats nouveaux qui y figurent sont dus à ce dernier; notre contribution est uniquement de nature rédactionnelle.

Neither scissors nor glue to laboriously prepare a 'final' manuscript (which, above all, must not reveal anything about the process that led to it)...

Récoltes et Semailles, March 1984, p. 1, note 1

"A NEW STYLE"

When he began publishing *Réflexions mathématiques* in 1983, Grothendieck chose to employ a "**renewed style**" of expression. He felt that his ideas had not been conveyed either through his writings, due to the formal style he used, or through his oral seminar **presentations** or "**private communications**," which would be lost to memory.



27.5.

60. It has been over six weeks now that I did'nt write down any notes. The reason for this is that I felt the story of usphericity structures and canonical modelizers was going to come now without any problem, almost as a matter of routine tox write it down with some care – therefore I started doing some scratchwork on a few questions which had been around but kept in the background since the beginning, which were a lot less clear in my mind. Some reflection ~~on these~~ was needed anyhow, before it would make much sense to start writing down anything on these. Finally it took longer than expected, as usual,– partly because (as usual too) a few surprises would turn up on my way. Also, I finally allowed myself to become distracted by some reflection on the"Lego-Teichmüller construction game", ~~especially~~ and pretty much so during last week. The occasion was a series of talks unformal talks Y. has been giving in Molino's seminar, on Thurston's ~~xxxxxxxxx~~ hyperbolic geometry game and his compactification of Teichmüller space.

Grothendieck's new style is notably visible through his descriptions of his daily mathematical activities. *Pursuing Stacks*, p. 191.
Archives Grothendieck, © Université de Montpellier

> *From now on, it would no longer be a question of presenting meticulous, well-groomed foundations for some new mathematical universe in the making. Rather, it would be "logbooks" in which the work would continue on a day-to-day basis, without hiding anything and as it really happens, with its failures and blunders, its insistent backward steps and sudden leaps forward – work that is irresistibly pulled forward day after day (despite countless incidents and unforeseen events), as if by an invisible thread – by some elusive, tenacious and sure vision.*
>
> *Récoltes et Semailles*, "Une lettre," May 1985, p. L17

Nevertheless, Grothendieck did not reject the classical axiomatic-deductive style of mathematical writing, which he continued to practice and teach. He simply wanted to reveal to his readers the process of discovery and the crucial and positive role of errors hidden by "**formal writing**."

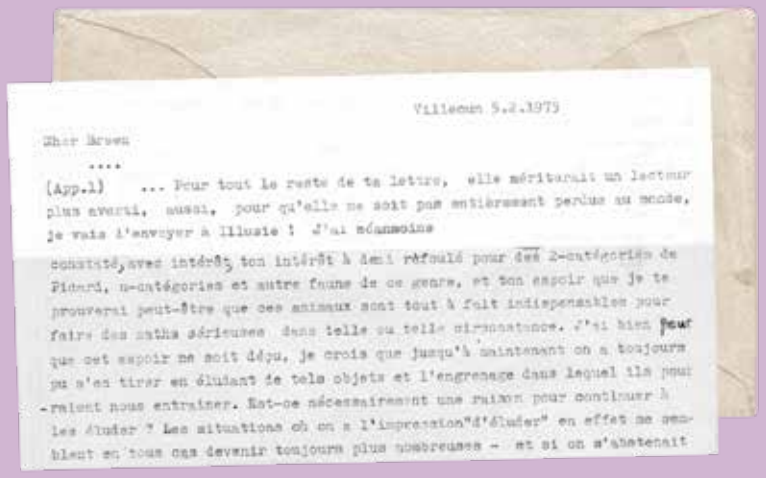
His new writing practice, presented as spontaneous, was that of a "**logbook**." It aimed at conveying a passion for mathematics but was nevertheless based on material. Preparatory notes, the "scratchworks," were produced by Grothendieck between the actual writing phases but were not displayed.

> *But whatever I am going to write in mathematics, I believe a major part of it will be 'speculation' or 'fiction', going hand in hand with painstaking, down-to-earth work to get hold of the right kind of notions and structures, to work out comprehensive pictures of still misty landscapes. The notes I am writing up lately [Pursuing Stacks] are in this spirit [...]*
>
> Letter from Alexandre Grothendieck to Ronnie Brown, 14 June 1983

PURSUING STACKS

A THEME ROOTED IN THE PAST

The origins of *Pursuing Stacks* can be found in Grothendieck's cohomological work. During the 1960s, he felt the need for a synthesis between the homological and homotopical approaches, which would be achieved by a new science, "**topological algebra**," in the service of **arithmetic geometry**.



Villecun 5.2.1975

Cher Breen

....

(App.1) ... Pour tout le reste de ta lettre, elle mériterait un lecteur plus averti, aussi, pour qu'elle ne soit pas entièrement perdue au monde, je vais l'envoyer à Illusie ! J'ai néanmoins

constaté avec intérêt ton intérêt à demi refoulé pour des 2-catégories de Picard, n-catégories et autre faune de ce genre, et ton espoir que je te prouverai peut-être que ces animaux sont tout à fait indispensables pour faire des maths sérieuses dans telle ou telle circonstance. J'ai bien peur que cet espoir ne soit déçu, je crois que jusqu'à maintenant on a toujours pu s'en tirer en éludant de tels objets et l'engrenage dans lequel ils pour-raient nous entraîner. Est-ce nécessairement une raison pour continuer à les éluder ? Les situations où on a l'impression d'éluder" en effet me semblent en tous cas devenir toujours plus nombreuses - et si on s'abstenait

Letter to Breen, 5 February 1975,
Archives Grothendieck
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> *"This is a broad synthesis of ideas from homological algebra [...], homotopical algebra, the topos version of "general topology," and finally from the theory (in limbo since the 1960s) of (non-strict) ∞-categories, or, as I now prefer to say, ∞-stacks.*
>
> *Récoltes et Semailles*, 4 December 1984, p. 638



At that time, Grothendieck faced serious technical problems which stemmed from the notion of ***triangulated categories***. To overcome them, he considered the formalism of ***n-categories*** and ***n-stacks***.

Unable to "sell" his program to other mathematicians, Grothendieck eventually set out to work it out himself and put forward a "sketch of foundations" for homotopy theory in *Pursuing Stacks* in 1983.

SCATTERED DIALOGUES

Grothendieck explained at length the **motivations** and **geometric** issues regarding his program of topological algebra in a letter to **Larry Breen** of 1975.

From February 1982 onwards, he corresponded "in a rambling way" with **Ronnie Brown** about n-groupoids and **n-categories**. A year later, Grothendieck realized that the concept promoted by Brown was that **of strict ∞-groupoid**. As this did not allow him to model the whole **homotopy types** of spaces, he decided to pursue stacks on his own.

A few days later, Grothendieck wrote a letter to **Daniel Quillen**, which would form the first pages of *Pursuing Stacks*. In this letter, he outlined for the first time the definition of **a weak ∞-groupoid**.



Daniel Quillen (1940–2011),
author of *Homotopical Algebra* (1967),
a seminal text in the field

and we foresee a lot of crap looming up in giving a reasonable definition of a non-strict n-category...

Letter from Grothendieck to Breen dated 5 February 1975

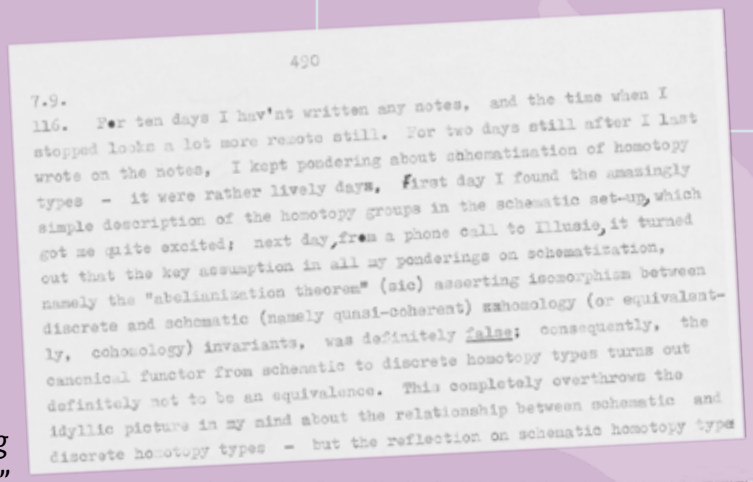
AN ENDLESS JOURNEY

Pursuing Stacks was a striking example of the new style of expression advocated by Grothendieck in *Récoltes et Semailles*. This "logbook of a **discovery journey**" completed between 19 February and 12 November 1983 sometimes turned into a personal diary.

Grothendieck cast himself in the role of an explorer and strived to convey his emotions to the reader, revealing the failures and unexpected events that punctuate **mathematical discovery**.

In the first part of *Pursuing Stacks*, "The Modelizing Story," numerous digressions interrupted the narrative of research devoted to models of homotopy types. This first, unfinished part, was the only one written and the subsequent volumes, which were to focus on stacks, never saw the light of day. The initial geometric vision revealed to Breen was therefore not concretized by Grothendieck.

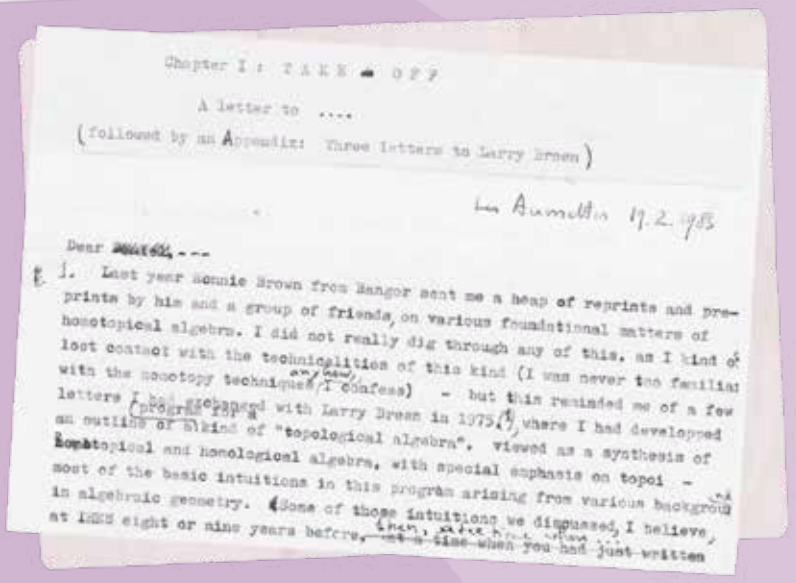
On a page dated 7 September 1983, Grothendieck commented on a telephone conversation with Luc Illusie. The latter invalidated an important hypothesis that Grothendieck was trying to prove and shattered an "idyllic image." Grothendieck did not return to this "*coup de théâtre*" until 26 September.



490

7.9.
116. For ten days I hav'nt written any notes, and the time when I stopped looks a lot more remote still. For two days still after I last wrote on the notes, I kept pondering about schematization of homotopy types - it were rather lively days, first day I found the amazingly simple description of the homotopy groups in the schematic set-up, which got me quite excited; next day, from a phone call to Illusie, it turned out that the key assumption in all my ponderings on schematization, namely the "abelianization theorem" (sic) asserting isomorphism between discrete and schematic (namely quasi-coherent) cohomology (or equivalently, cohomology) invariants, was definitely false; consequently, the canonical functor from schematic to discrete homotopy types turns out definitely not to be an equivalence. This completely overthrown the idyllic picture in my mind about the relationship between schematic and discrete homotopy types - but the reflection on schematic homotopy types

Pursuing Stacks, p. 490,
Archives Grothendieck
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Chapter I : TAKE - OFF

A letter to ....

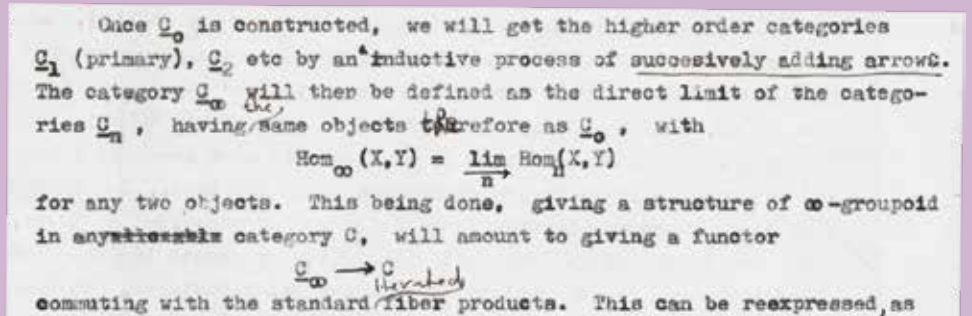
(followed by an Appendix: Three letters to Larry Breen)

Montpellier 19.2.83

Dear [redacted] ---

1. Last year Ronnie Brown from Bangor sent me a heap of reprints and preprints by him and a group of friends, on various foundational matters of homotopical algebra. I did not really dig through any of this, as I kind of lost contact with the technicalities of this kind (I was never too familiar with the homotopy techniques, I confess) - but this reminded me of a few letters I had exchanged with Larry Breen in 1975/6, where I had developped an outline of a kind of "topological algebra", viewed as a synthesis of homotopical and homological algebra, with special emphasis on topoi - most of the basic intuitions in this program arising from various background in algebraic geometry. (Some of those intuitions we discussed, I believe, at IHES eight or nine years before, [redacted] a time when you had just written

The first page of *Pursuing Stacks*: the letter to Quillen dated 19 February 1983,
Archives Grothendieck © Université de Montpellier



Once C_0 is constructed, we will get the higher order categories C_1 (primary), C_2 etc by an inductive process of successively adding arrows. The category C_∞ will then be defined as the direct limit of the categories C_n, having same objects therefore as C_0, with

$$\mathrm{Hom}_\infty(X,Y) = \varinjlim_n \mathrm{Hom}_n(X,Y)$$

for any two objects. This being done, giving a structure of ∞-groupoid in any category C, will amount to giving a functor

$$C_\infty \longrightarrow C$$

commuting with the standard fiber products. This can be reexpressed, as

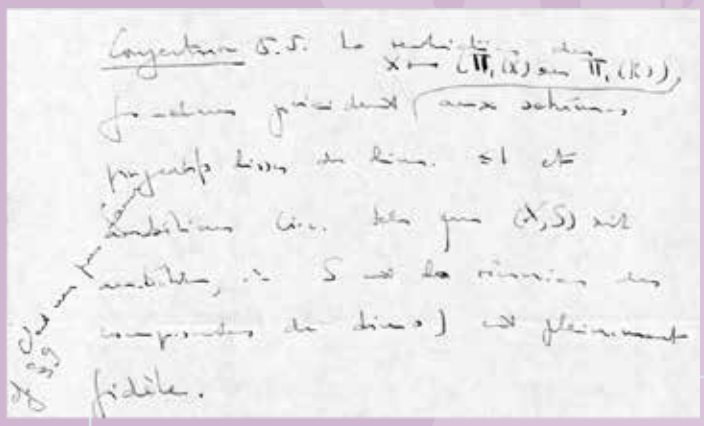
The definition of *weak ∞-groupoid* in the letter to Quillen, *Pursuing Stacks*, p. 9,
Archives Grothendieck
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GALOIS-TEICHMÜLLER THEORY

THE LONG MARCH THROUGH GALOIS THEORY

Grothendieck laid the foundations for **anabelian geometry** and **Galois-Teichmüller theory** in *La Longue Marche à travers la théorie de Galois*. This 1,300-page manuscript was written between January and June 1981 during a "long period of **mathematical frenzy**."

This text, which was not intended for publication, already revealed some of the features of the new style of expression that would make its public appearance with *Pursuing Stacks* two years later. Grothendieck incorporated **speculative comments** and corrections for himself within a classical **axiomatic-deductive presentation**.



Grothendieck stated the main conjecture of his "anabelian geometry," which partially remains open today. In the margin, we read "This is a little wrong, see §9." In the following pages, he returned to this conjecture to refine it and correct its hypotheses.

La Longue Marche à travers la théorie de Galois, p. 9, verso, Archives Grothendieck © Université de Montpellier

A NEW MATHEMATICAL DREAM

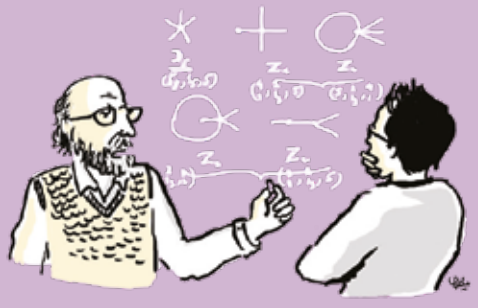
Grothendieck wanted to give substance to a "**dream**" that resounded with that of **motives**. It began in 1977 in the wake of the "*Grothendieck-eries*" on the topology of surfaces in connection with Yves Ladegaillerie's thesis. In June 1978, he presented his ideas on "***dessins d'enfants***" to **Pierre Deligne** and **Barry Mazur** during a visit to IHES with **Jean Malgoire**.

Grothendieck decided to embark on the *Long March* three years later, having found no one among his students or colleagues to undertake the "exploration journey of this 'new world'." This was a characteristic ***leitmotif*** of his time in Montpellier.

> *"[…] 'anabelian algebraic geometry', which originated in a study […] of the action of 'absolute' Galois groups […] on (profinite) fundamental geometric groups of algebraic varieties […], and more particularly (breaking with a well-established tradition) fundamental groups that are very far from abelian groups (and which I therefore call 'anabelian').*
>
> *Esquisse d'un Programme*, January 1984, p. 14

The ultimate outcome of Galois theory, and most probably in the spirit of Galois

Récoltes et Semailles, February 1984, p. 16

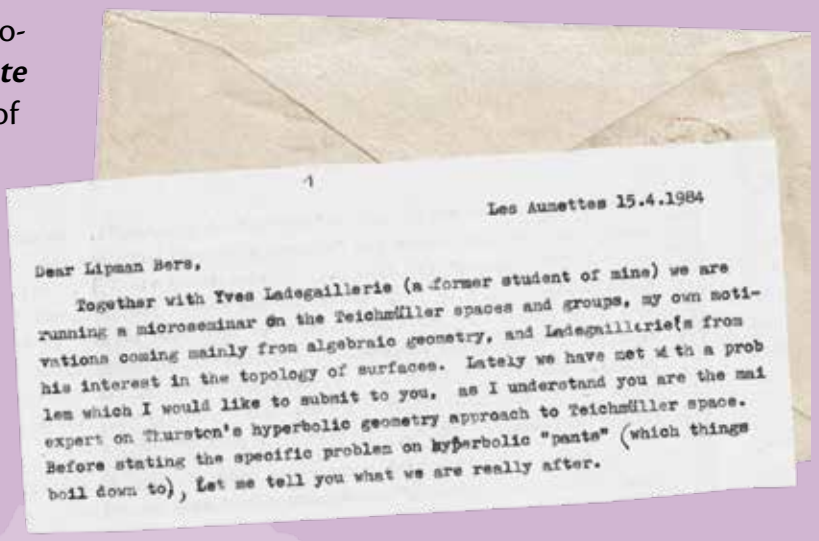


THE GALOIS-TEICHMÜLLER YOGA

Within the **anabelian geometry** program, Grothendieck quickly chose to focus on ***the absolute Galois group*** action on ***Teichmüller's*** "tower" of ***groupoids***, "the most fascinating object [he] had encountered in mathematics."

In the spring of 1983, seminar presentations given by his student **Yves Ladegaillerie** led him to discover links between his research and **William Thurston**'s work on **hyperbolic geometry** about Teichmüller spaces.

Letter to Lipman Bers, 15 April 1984
Archives Grothendieck © Université de Montpellier



Les Aumettes 15.4.1984

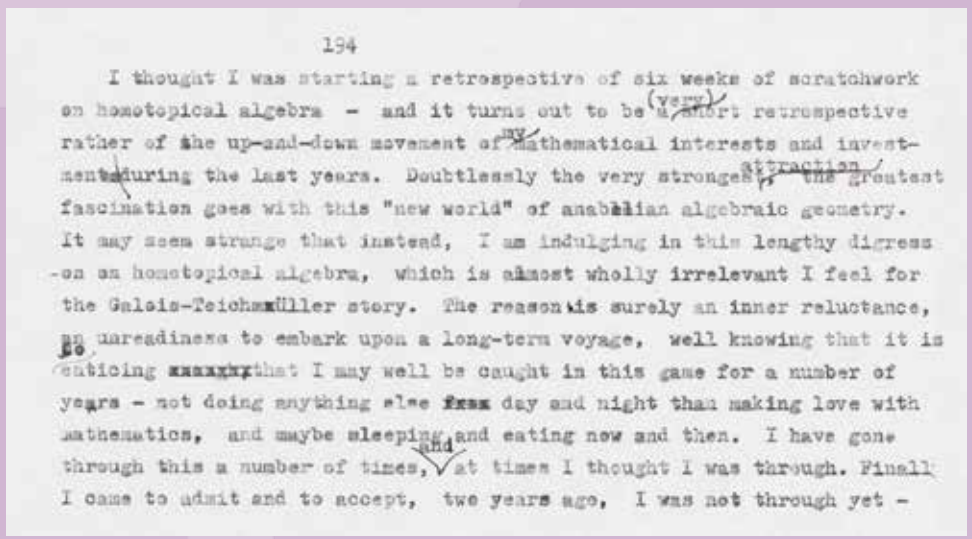
Dear Lipman Bers,

Together with Yves Ladegaillerie (a former student of mine) we are running a microseminar on the Teichmüller spaces and groups, my own motivations coming mainly from algebraic geometry, and Ladegaillerie's from his interest in the topology of surfaces. Lately we have met with a problem which I would like to submit to you, as I understand you are the main expert on Thurston's hyperbolic geometry approach to Teichmüller space. Before stating the specific problem on hyperbolic "pants" (which things boil down to), let me tell you what we are really after.

JUMPING INTO THE UNKNOWN?

The fascination with the themes of this second **"crucial" part** of **arithmetic geometry** never waned, as witnessed by exchanges of **letters**, for example with **Gerd Faltings** and **Lipman Bers**, and the organization of **seminars** devoted to the "Teichmüller tower" between 1981 and 1984.

A hundred pages of notes and scratchworks dealt with Galois-Teichmüller yoga. However, the part of *Réflexions mathématiques* that was to be devoted to anabelian geometry was never written. This should undoubtedly be seen as a sign of Grothendieck's inner resistance to a new long-term mathematical journey.



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I thought I was starting a retrospective of six weeks of scratchwork on homotopical algebra - and it turns out to be a (very) short retrospective rather of the up-and-down movement of my mathematical interests and investment during the last years. Doubtlessly the very strongest, the greatest attraction, fascination goes with this "new world" of anabelian algebraic geometry. It may seem strange that instead, I am indulging in this lengthy digression on homotopical algebra, which is almost wholly irrelevant I feel for the Galois-Teichmüller story. The reason is surely an inner reluctance, an unreadiness to embark upon a long-term voyage, well knowing that it is so enticing that I may well be caught in this game for a number of years - not doing anything else from day and night than making love with mathematics, and maybe sleeping and eating now and then. I have gone through this a number of times, and at times I thought I was through. Finally I came to admit and to accept, two years ago, I was not through yet -

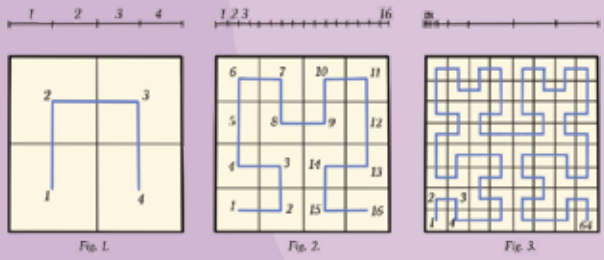
Pursuing Stacks, p. 194
Archives Grothendieck
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TOPOLOGIES

FAKE PROBLEMS AND WILD PHENOMENA

In *Esquisse d'un Programme*, Grothendieck observed that **general topology** was developed for the purposes of analysis and was unsuited to algebraic geometry. Fake problems and **wild phenomena** reflected this issue regarding the foundations of topology.

A continuous plane curve that filled a square was described by **David Hilbert** in an article of *Mathematische Annalen* in 1891. This is one of the most famous examples of these wild phenomena, in contrast to the intuition, which accompanied the birth of set theory.



The Hilbert curve defined by an iterative process, *Mathematische Annalen* (1891)

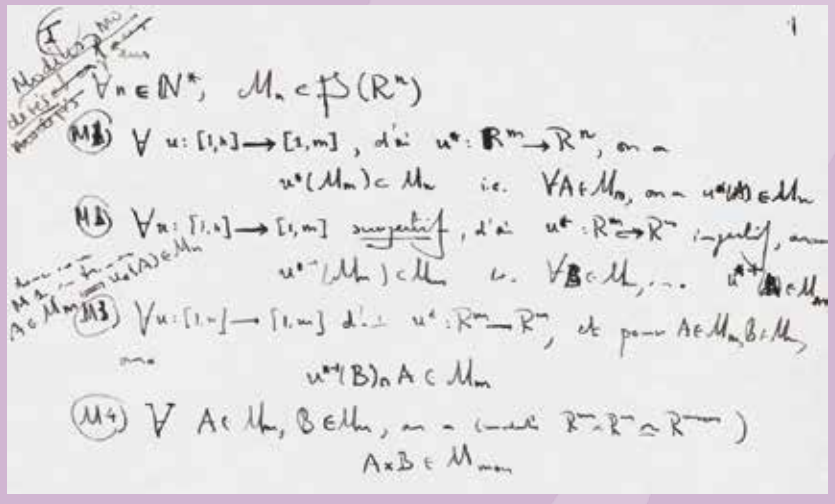
> *"This inadequacy of the foundations of topology manifested itself from the outset in 'fake problems' (at least from the point of view of the topological intuition of shapes) [...] Even today, as in the heroic days when we first saw, with concern, curves cheerfully filling squares and cubes, when one set out to do topological geometry in the technical context of topological spaces, we encountered at every step parasitic difficulties related to wild phenomena.*
>
> *Esquisse d'un Programme*, January 1984, p. 29

A TAME TOPOLOGY

The "heuristic reflections" concerning a so-called tame topology "capable of easily expressing **the topological intuition of shapes**" occupied a dozen pages of *Esquisse d'un Programme*.

Grothendieck's approach to build a new "**geometric topology**" seemed to echo the considerations of his elder and friend **Claude Chevalley** on the axiomatic style.

One finds some traces of Grothendieck's attempts **at axiomatization** in two Montpellier manuscripts of less than a hundred pages but no lengthy developments like those of the *Long March*. This theory was nonetheless supposed to produce a second "metamorphosis of the notion **of space**" after that accomplished by topos.



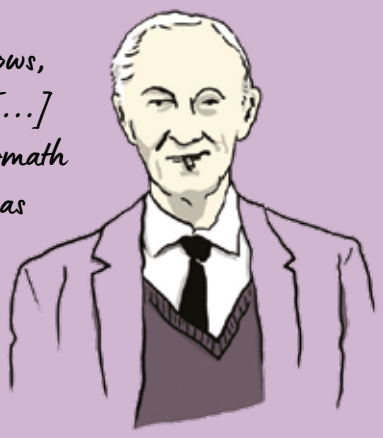
An early version of the definition of "*tame structure*" [1973-1978?]
Archives Grothendieck © Université de Montpellier

Down with so-called 'general' topology...

Esquisse d'un Programme, January 1984, Sommaire

"[...] the constitution of domains precisely adapted to certain operations allows, within these domains, general statements to be made about the objects being studied. [...] Thanks to this, the mathematician has no longer the impression, as he did in the aftermath of the discovery of non-differentiable functions and other apparent oddities, that he was progressing on ground strewn with pitfalls; on the contrary, he finds himself in the presence of strongly structured theories in which some approaches can be taken with complete freedom, while others are excluded because they lose all meaning.

Claude Chevalley, *Variations du style mathématique* (1935), p. 382.

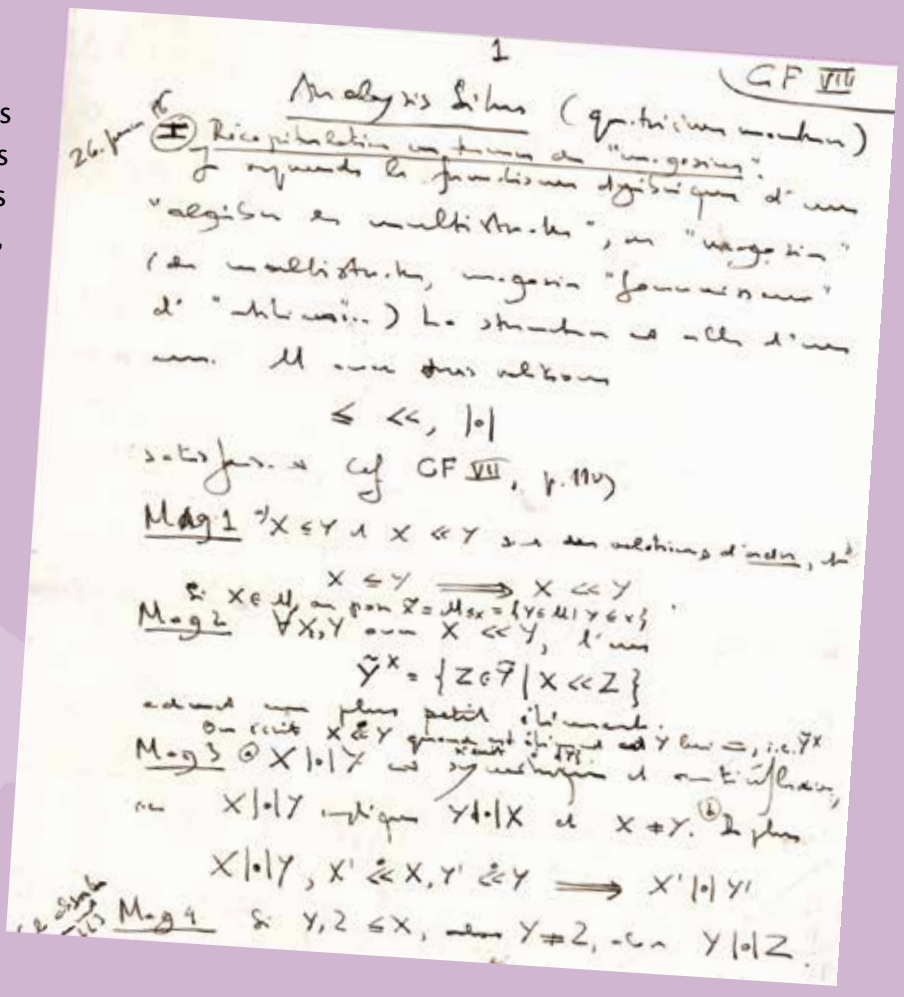


Claude Chevalley
(1909-1914)

TOWARDS A GEOMETRY OF SHAPES

Vers une Géométrie des formes was written over a period of several weeks between 5 June and 15 July 1986. In this overlooked text of nearly 700 pages, Grothendieck proposed his last vision of topology and **shape** without really revealing his motivations.

He used an axiomatic approach. The **combinatorial** perspective adopted was elementary and suitable for studying classical figures such as **polyhedra**, which were privileged objects of thinking during the Montpellier period. A litany of axioms, constantly revised in the four successive versions of an ***Analysis situs***, defined new concepts with unexpected names: *"ateliers," "contrées," "disjoncteurs," "magasins," "spatios,"* etc.



Axioms of "magasins," *Vers une Géométrie des formes* (1986), VII. Analysis situs (fourth draft), p. 1,
Archives Grothendieck © Université de Montpellier

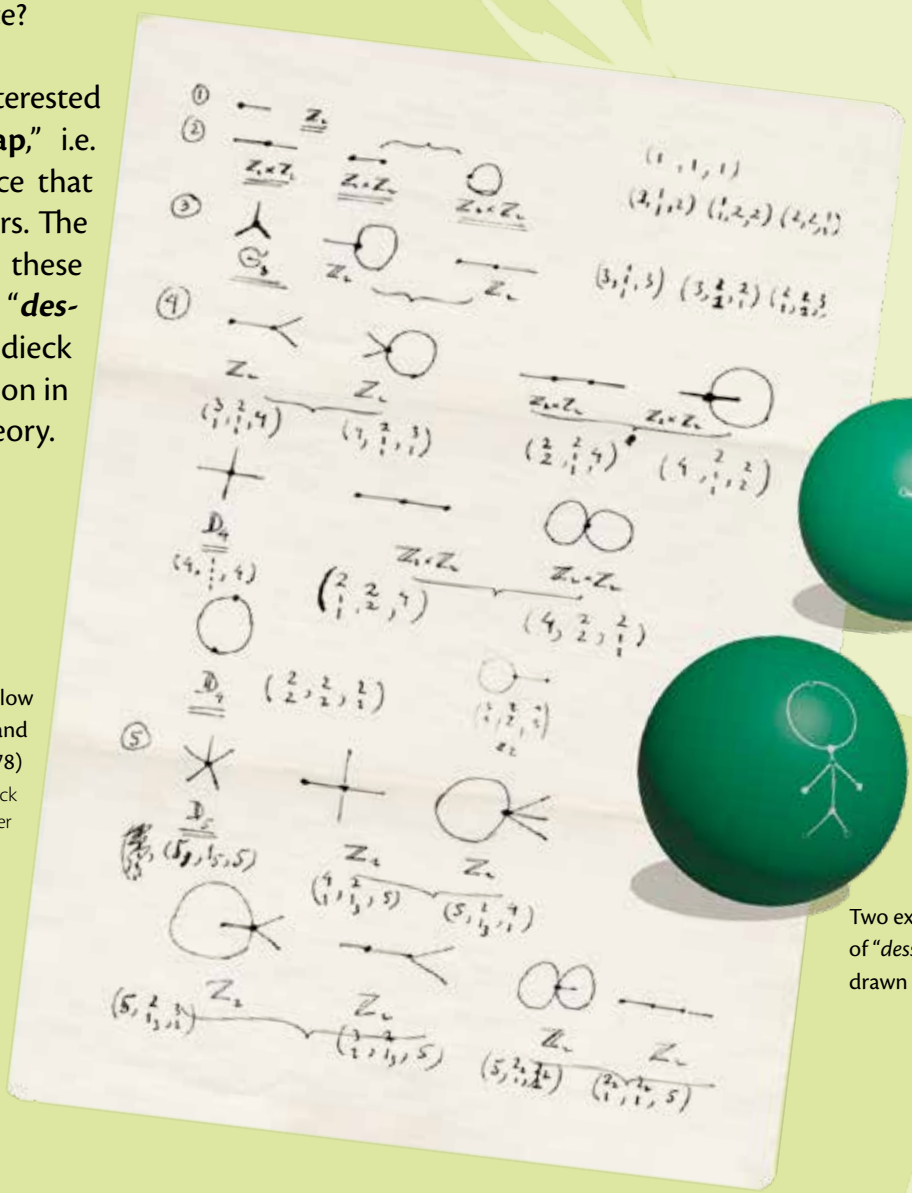
FROM THE TOPOLOGY OF SURFACES TO *DESSINS D'ENFANTS*

> *"I don't think any mathematical fact has ever struck me as much as this one or had a comparable psychological impact. This is surely due to the very familiar, non-technical nature of the objects considered.*
>
> *Esquisse d'un Programme*, January 1984, p. 12

MAPS

Grothendieck studied **surfaces** with his students from a purely **topological** point of view and was interested in questions of graph embeddings. For example, how many ways are there to draw a given graph on a given surface?

He was particularly interested in the notion of **a** "**map**," i.e. graphs drawn on a surface that divide it into simple sectors. The visual representation of these graphs is reminiscent of "***dessins d'enfants***." Grothendieck retained this vivid expression in the terminology of the theory.



Grothendieck explores "low degree" *dessins d'enfants* and their symmetries (1976-1978)
Archives Grothendieck
© Université de Montpellier

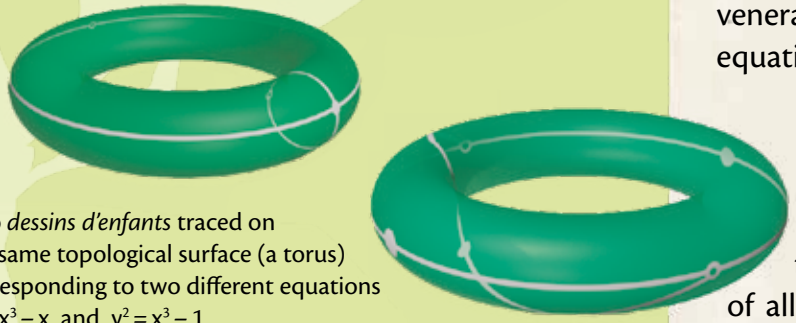
Two examples of "*dessins d'enfants*" drawn on a sphere

... of which any child's drawing scribbled on a piece of paper ... provides a perfectly explicit example.

Esquisse d'un Programme, January 1984, p. 12

DESSINS D'ENFANTS

Grothendieck and his students then noticed that a *dessin d'enfant* drawn on a surface could be used to reconstruct an equation of this surface, which is now seen as a **Riemann surface**.



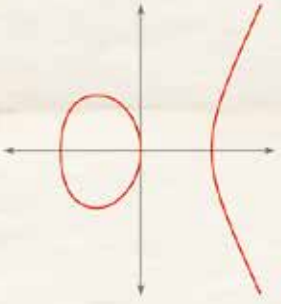
Two *dessins d'enfants* traced on the same topological surface (a torus) corresponding to two different equations $y^2 = x^3 - x$ and $y^2 = x^3 - 1$

Beneath their naive appearance, ***dessins d'enfants*** are thus mathematical objects of great depth, that reveal a new and unexpected connection between the algebraic world of equations and the topological world of surfaces. Indeed, several equations may express the same surface. Being able to progress "in reverse" is a truly remarkable fact. This discovery had a profound effect on Grothendieck.

The set of solutions of an algebraic equation in two variables f(x,y) = 0 is classically represented, in the real Cartesian plane, as a curve formed by the set of points (x,y) with real coordinates satisfying the equation f(x,y)=0. Just think of the venerable conics that correspond to quadratic equations.

However, such a curve represents only a small part of the solutions of the equation. The latter has a multitude of complex solutions as well as "infinite" solutions. The set of all these solutions forms a two-dimensional geometrical extension called a **Riemann surface**.

The real solutions are represented there in the form of a one-dimensional curve.



The real solutions of the equation $y^2 = x^3 - x$ represented in Cartesian coordinates



In green, the complex solutions of the equation $y^2 = x^3 - x$; in red, its real solutions



Here, the mathematical reality is a little more complex. More precisely, a *dessin d'enfant* corresponds to an equation **and** a particular rational fraction called ***a Belyi function***.

FROM *DESSINS D'ENFANTS* TO GALOIS THEORY

BELYI'S THEOREM

It was during informal discussions with his students that Grothendieck discovered a remarkable arithmetic property of the algebraic equations associated with *dessins d'enfants*: their coefficients are **algebraic numbers**. He then put forward a hypothesis that he himself described as "crazy."

This hypothesis predicted that all algebraic equations whose coefficients are algebraic numbers could be associated with a *dessin d'enfant*. It thus suggested a profound link between **Galois theory** and *dessins d'enfants*.

Submitted to "the experts in the field," this hypothesis was cautiously received until it was confirmed by the Soviet mathematician **Gennadii Vladimirovich Belyi** in 1978 by means of a very simple demonstration.



Gennadii Vladimirovich Belyi
(1951–2001)

TOWARDS THE GROTHENDIECK-TEICHMÜLLER THEORY

The **absolute Galois group**, which by definition operates on algebraic numbers, can therefore be conceived as a group of transformations of *dessins d'enfants*. The theory of *dessins d'enfants*, originally motivated by the topological study of surfaces, thus achieved a spectacular connection between topology and Galois theory. This theory was developed by Grothendieck in *La Longue Marche à travers la théorie de Galois*, written in 1981.

The group of transformations of the field $\overline{\mathbb{Q}}$ of algebraic numbers is called **the absolute Galois group**. It is denoted by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Its elements are permutations that transform one algebraic number into another algebraic number, while preserving addition and multiplication. An algebraic number that is a solution of an equation $P(x)=0$ is also transformed into another solution of the equation. A fundamental example is that of complex conjugation, which transforms the number i into $-i$. The absolute Galois group constitutes the mysterious core of number theory. Despite nearly two centuries of effort, its structure has still not been fully elucidated.

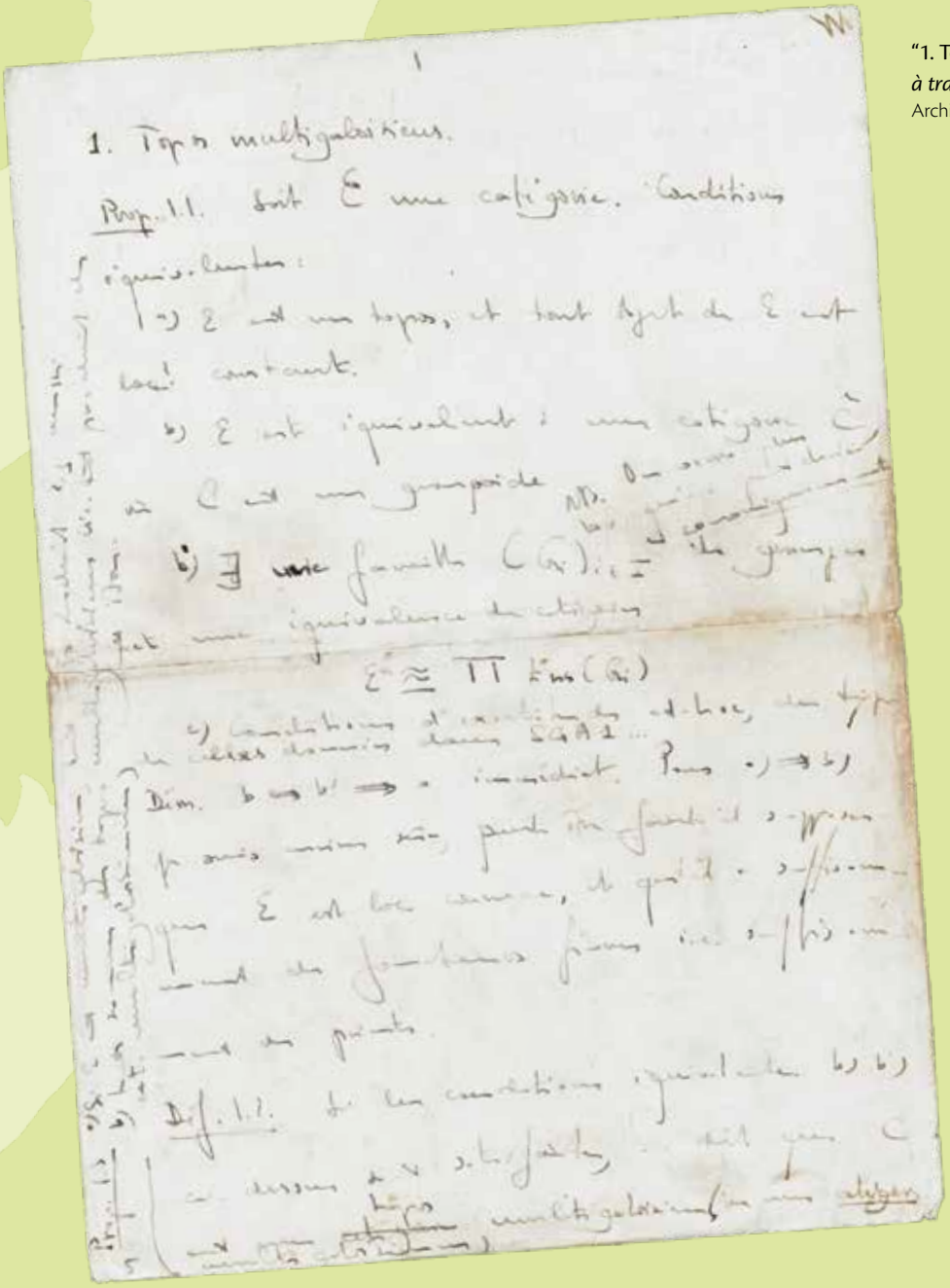


A *dessin d'enfant* and two new *dessins* obtained by using transformations of the Galois group.

This shows that it is still possible to prove 'extraordinary' things at little cost.

Letter from Grothendieck to Malgoire, 25 August 1978

A number which is the solution of a polynomial equation with rational coefficients is called an ***algebraic number***. $\sqrt{2}$ (solution of $x^2-2=0$), the golden ratio φ (solution of $x^2-x-1=0$), the imaginary number i (solution of $x^2+1=0$), or the roots of unity $e^{2i\pi/n}$ (solutions of $x^n-1=0$) are well-known examples of algebraic numbers. The numbers e and π, on the other hand, are not solutions of an algebraic equation with rational coefficients: they are called ***transcendental numbers***. Algebraic numbers form a field denoted by $\overline{\mathbb{Q}}$.



"1. Topos multigaloisiens," *La Longue Marche à travers la théorie de Galois* (1981), p. 1, Archives Grothendieck © Université de Montpellier

THE GROTHENDIECK-TEICHMÜLLER THEORY

The moduli space of the straight lines in the plane passing through the origin

In order to understand the profound nature of the action of the absolute Galois group on *dessins d'enfants*, Grothendieck undertook to study the permutation of Riemann surfaces by such an action. These were classically organized into a space studied by **Oswald Teichmüller** in the late 1930s. The resulting geometric Galois theory is now called "**Grothendieck-Teichmüller theory**."

MODULI SPACES OF RIEMANN SURFACES

The main objects in this theory are the spaces $\mathbf{M_g}$ which parameterize all **Riemann surfaces** of genus g.

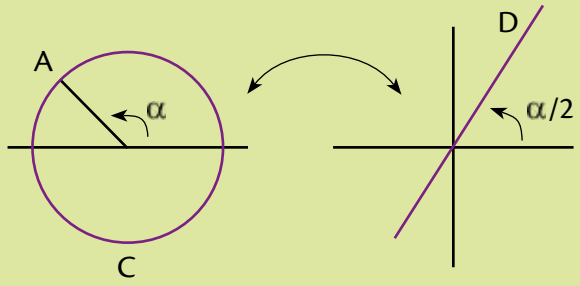
The ***genus*** of a surface measures its topological complexity by counting its "number of handles."



genus 0 genus 1 genus 2

Surfaces of genus 0, 1, 2

A ***moduli space*** is a space whose points describe geometric objects or configurations. For example, the moduli space of lines in the plane passing through the origin is a circle: every point A on the circle C, marked by the angle α corresponds to one and only one line D in the plane passing through the origin and forming an angle $\beta = \alpha/2$ with the horizontal.



The points of these spaces are in one-to-one correspondence with Riemann surfaces. The spaces M_g provide a classic example of ***moduli spaces***.

The absolute Galois group therefore permutes Riemann surfaces by transforming the equations of the corresponding *dessins d'enfants*. In doing so, it acts on the geometry of the spaces M_g. This geometric action is at the heart of **Grothendieck-Teichmüller theory** and, conjecturally, allows a complete description of the absolute Galois group.

I got involved a bit with recalling to mind the 'Lego-Teichmüller construction game' for describing in a concrete, 'visual' way the Teichmüller groups...

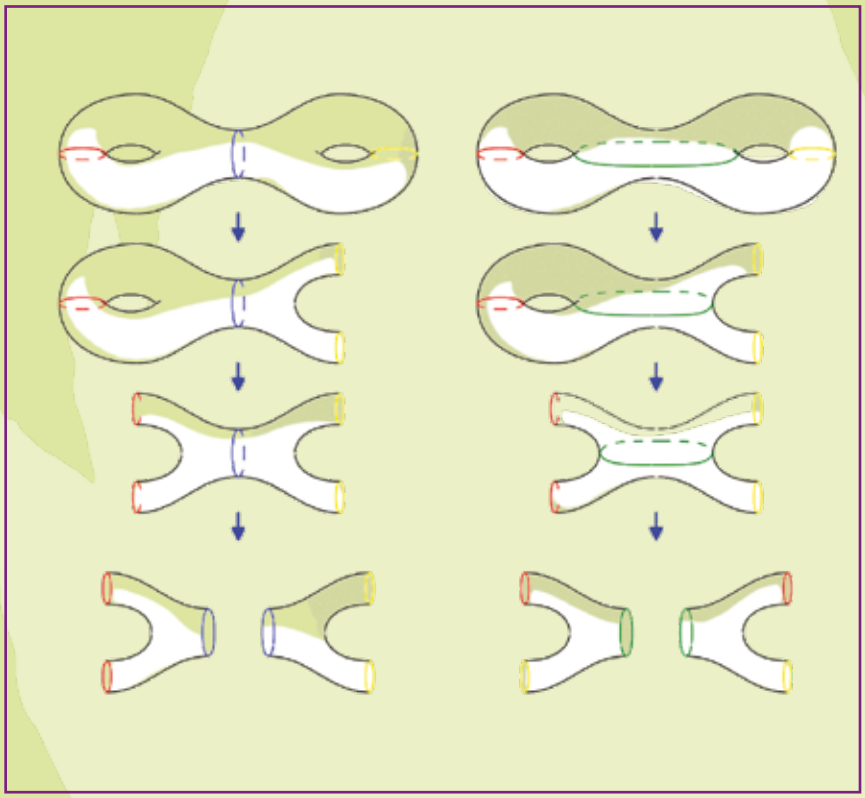
Pursuing Stacks, 14 March 1983, p. 36.

A LEGO SET

The spaces M_g remain interconnected when the genus g varies thanks to a process of "**pants decomposition**." This is a simplification operation that allows one to understand surfaces of a given genus using those of a lower genus. Grothendieck used the term "**Lego-Teichmüller set**" and predicted that understanding the absolute Galois group requires only the study of the spaces M_0 and M_1.

The symmetries of this Lego set form the **Grothendieck-Teichmüller group**, named by **Vladimir Drinfeld**, who introduced it in 1991. Surprisingly, this group also plays a role in **theoretical physics**.

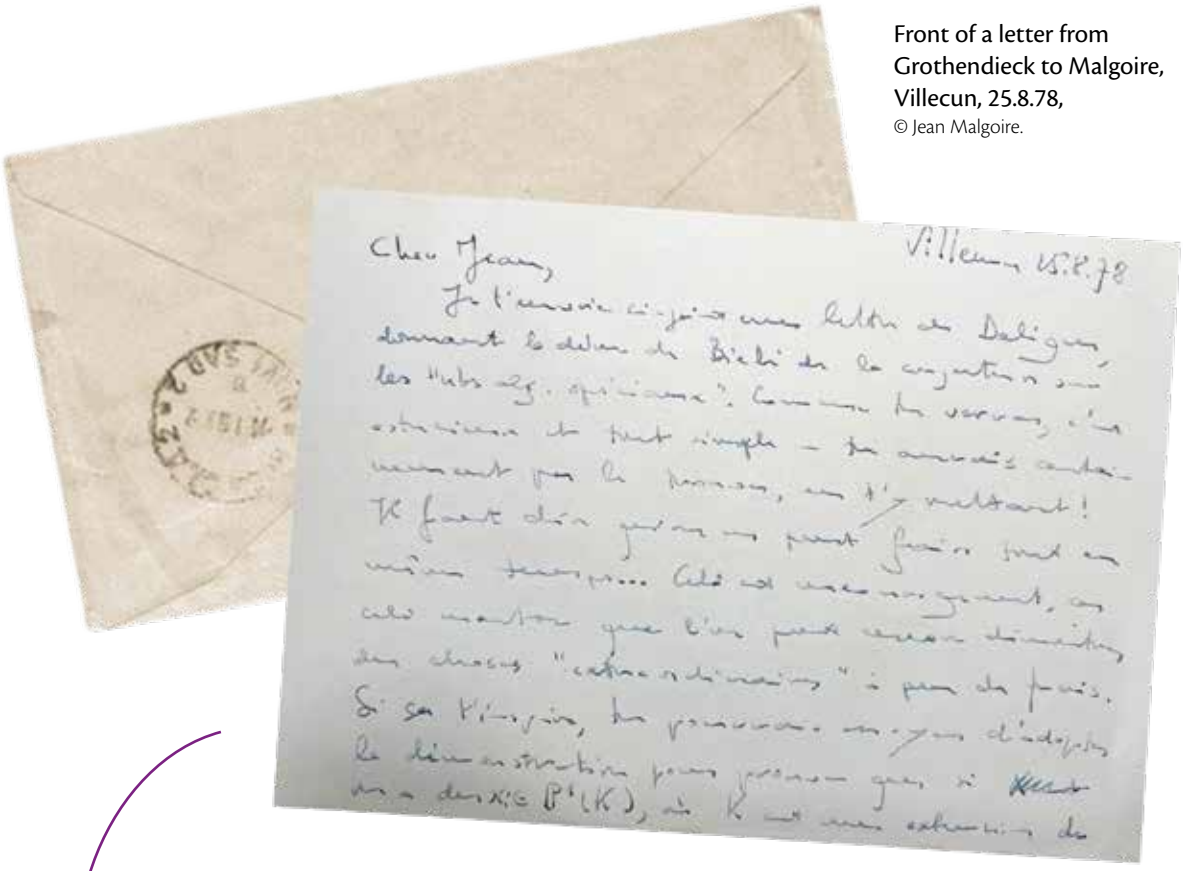
Two pants-shaped cut-outs from a surface of genus 2



"

The increasingly obvious relationship between the Grothendieck-Teichmüller group on the one hand, and the renormalisation group of quantum field theory on the other, may be the first manifestation of a symmetry group of the fundamental constants of physics, a kind of cosmic Galois group!

Pierre Cartier, "*Un pays dont on ne connaîtrait que le nom (Grothendieck et les motifs)*" (2000), p. 21.



Front of a letter from Grothendieck to Malgoire, Villecun, 25.8.78,
© Jean Malgoire.

Villecun, 25 August 1978

Dear Jean,

I am sending you a letter from Deligne, giving Bieli's [Belyi's] proof of the conjecture on "special algebraic numbers." As you will see, it is ingenious and very simple – you could certainly have found it yourself if you had tried! It must be said that one cannot do everything at once... This is encouraging, because it shows that we can still discover "extraordinary" things at little cost. If it inspires you, you could try to adapt the proof to show that if we have $x_i \in \mathbb{P}^1(K)$, where K is an extension of transcendence degree n of its prime fields ($\mathbb{Q}$ if the characteristic is zero), $1 \leqslant i \leqslant N$, then $\exists f : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ which is unramified outside a set S of $n+3$ points, and with $f(x_i) \in S$, $\forall i$. Hence: any algebraic curve defined over K is a [covering] of $\mathbb{P}^1_K$ unramified outside $n+3$ points.

Kind regards

Alexandre Grothendieck

A populated solitude

"But I know full well [...] that the things that interest me cannot fail to interest someone, or even several people, whether in ten years or a hundred years, it doesn't really matter. That's what gives meaning to my work, even if it is done in solitude [...] Mathematics is a collective adventure that has been going on for thousands of years."

Alexandre Grothendieck, ***Récoltes et Semailles***, February 1984, pp. 126-127

In January 1984, Alexandre Grothendieck had "two large boxes of handwritten notes" under his arm, containing several thousand pages of mathematical reflections, which he referred to in *Esquisse d'un Programme*. This text was addressed to the National Committee to justify an application for admission to the CNRS, but also to a few mathematicians who read it and passed it on to others. The application was successful and Grothendieck returned to his position as a researcher at the CNRS until his retirement in 1988. He withdrew to Lasserre, in Ariège, in 1991 and died in 2014. He did not publish his *Réflexions mathématiques*. Is that the end of the story?

Grothendieck's ideas circulated through texts that were passed on to the mathematical community, such as *Pursuing Stacks*. They were assimilated and developed during the last thirty years. The notes accumulated over the years are part of this mathematical legacy. Recently, unpublished manuscripts from the period 1987-1999 have been added to the BnF's collections, complementing those from the period 1949-1991 available on the website of the Grothendieck Archives at the University of Montpellier. Who will read these texts?

One might be surprised at the interest shown by mathematicians in forgotten manuscripts, even if they are the work of a solitary genius. Contemporary mathematics, which is constantly advancing, has little to do with the erudition that is the hallmark of obsolescence in science, or the romanticism of a Chateaubriand who saw Pascal as a "frightening genius." Today, on an "open archive" website called arXiv, mathematicians from around the world enrich the body of mathematics with new elements every day. They write to be read and for their work to be discussed and used. This was also Grothendieck's intention. The apparent solitude of genius is destined to be populated by those who pursue the collective adventure of mathematics by reading his texts.

SOURCES (1973-1991)

- *Archives Grothendieck*, IMAG, University of Montpellier, https://grothendieck.umontpellier.fr
- *La Longue Marche à travers la théorie de Galois*, 1981, Archives Grothendieck, cote n° 140. Volume 1, §1-37, edited by J. Malgoire, Montpellier, University of Montpellier II, 1995.
- *Pursuing Stacks*, 1983, cote n° 134. Vol. 1, chaps. 1-4, edited by G. Maltsiniotis, Paris, Société Mathématique de France, 2022.
- *Esquisse d'un Programme*, 1984, Archives Grothendieck, cote n° 119. Edited by L. Schneps and P. Lochak in *Geometric Galois Actions, 1. Around Grothendieck's Esquisse d'un Programme*, Cambridge, Cambridge University Press, 1997, pp. 5-48.
- *Récoltes et Semailles*, 1984-1986, Archives Grothendieck, cote n° 170-177. Paris, Gallimard, Tel collection, "*version augmentée*," 2023.
- *Vers une Géométrie des formes*, 1986, Archives Grothendieck, cote n° 156.
- *Les Dérivateurs*, 1990-1991, Archives Grothendieck, cote n° 157. Manuscript currently being edited by M. Künzer, J. Malgoire, and G. Maltsiniotis, https://webusers.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html

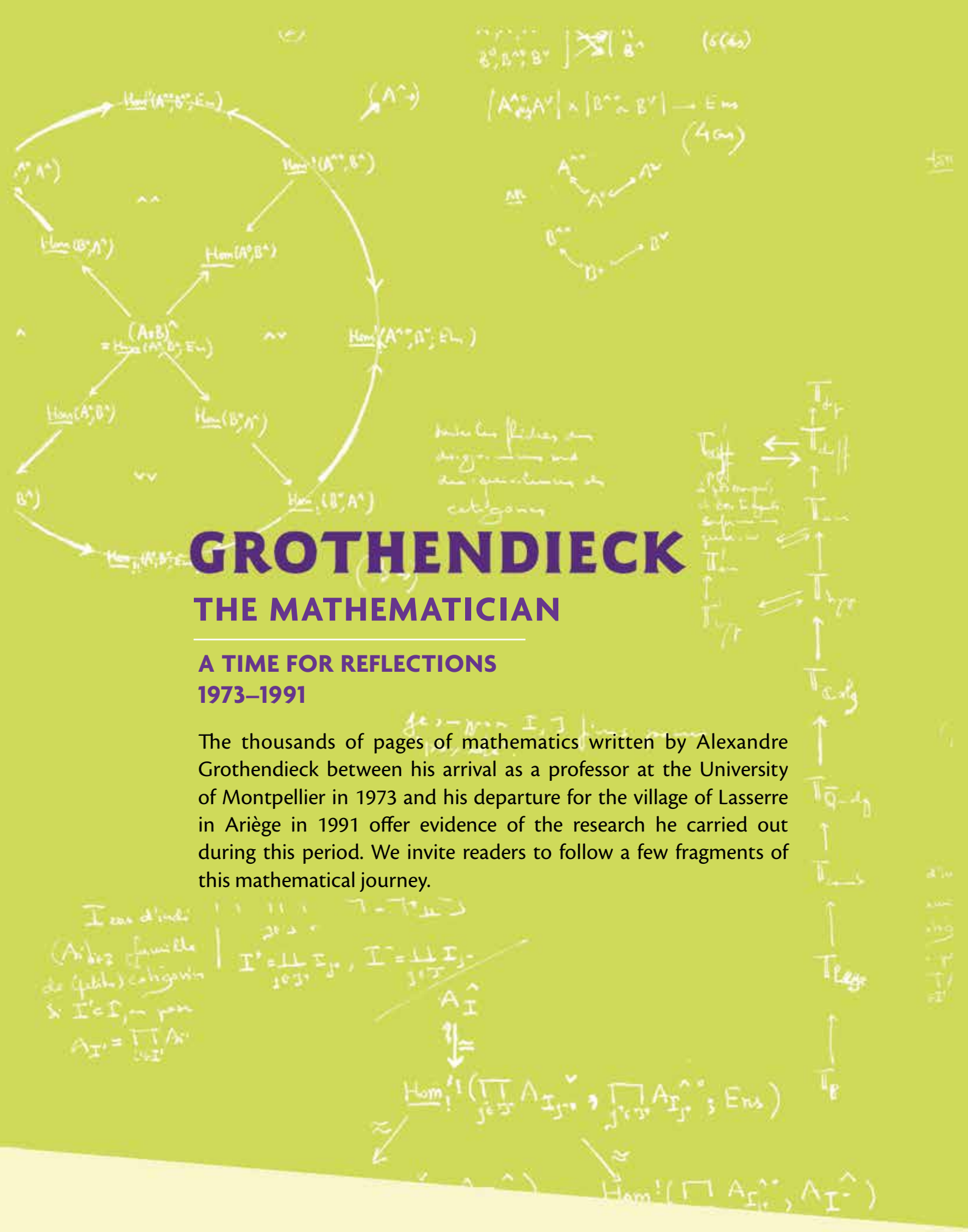
OTHER SOURCES CITED

- G.V. Belyi, "On Galois extensions of a maximal cyclotomic field," *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2), 1979, pp. 267-276; *Math. USSR-Izv.*, 14(2), 1980, pp. 247-256.
- A. Borel and J.-P. Serre, "The Riemann-Roch Theorem," *Bulletin of the SMF*, vol. 86, 1958, pp. 97–136. http://www.numdam.org/item?id=BSMF_1958__86__97_0
- C. Chevalley, "Variations du style mathématique," *Revue de métaphysique et de morale*, vol. 42(3), 1935, pp. 375-384.
- A. Grothendieck, "Standard conjectures on algebraic cycles," in *Algebraic Geometry, Papers presented at the Bombay Colloquium 1968*, Tata Institute of Fundamental Research, Bombay, Oxford University Press, 1969, pp. 193–199.
- A. Grothendieck, *Correspondance Grothendieck-Serre*, Paris, Société Mathématique de France, 2001.
- D. Hilbert, "Über die stetige Abbildung einer Linie auf ein Flächenstück," *Mathematische Annalen*, vol. 38, 1891, pp. 459–460.
- A. Weil, "Number of solutions of equations in finite fields," *Bull. Amer. Math. Soc.*, 55(5), 1949, 497-508.
- A. Weil, *Œuvres scientifiques*, vol. I (1926–1951), corrected second printing, New York, Springer Verlag, 1980.

SOME STUDIES

- P. Cartier, "Un pays dont on ne connaîtrait que le nom (Grothendieck et les 'motifs')," *Notes sur l'histoire et la philosophie des mathématiques IV*, IHES, November 2000, pp. 2-33.
- A. Jackson, "Comme appelé du néant – as if summoned from the void: the life of Alexandre Grothendieck," *Notices Amer. Math. Soc.*, 51(9), 1038-1056 and 51(10), 1196-1212, 2004.
- F. Jaeck (ed.), *Lectures Grothendieckiennes*, Paris, SMF-Spartacus, 2021.
- M. Panza, D. C. Struppa, J.-J. Szczeciniarz (eds.), *The mathematical and philosophical legacy of Alexander Grothendieck*, Cham, Birkhäuser, 2025.
- L. Schneps (ed.), *Grothendieck Circle. A website devoted to the life and works of Alexandre Grothendieck*, https://www.grothendieckcircle.org

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GROTHENDIECK

THE MATHEMATICIAN

A TIME FOR REFLECTIONS
1973–1991

The thousands of pages of mathematics written by Alexandre Grothendieck between his arrival as a professor at the University of Montpellier in 1973 and his departure for the village of Lasserre in Ariège in 1991 offer evidence of the research he carried out during this period. We invite readers to follow a few fragments of this mathematical journey.